**Example 6** Find $\int \sqrt[3]{2x-3}\ x\,dx$

**Method 1:** Use simple $u$-substitution.
Step 1: Rationalize the integrand.

Let $u=\sqrt[3]{2x-3}$ or $u=(2x-3)^{\frac{1}{3}}$

$u^3=2x-3$

$3u^2\dfrac{du}{dx}=2$ and $dx=\dfrac{3u^2}{2}du$ (by implicit differentiation)

Also, $x=(u^3+3)/2$ (for the factor "$x$" in $\int \sqrt[3]{2x-3}\ x\,dx$

Step 2: Substitute $u$ for $\sqrt[3]{2x-3}$; $\dfrac{3u^2}{2}du$ for $dx$ in $\int \sqrt[3]{2x-3}\ x\,dx$.

Then $\int \sqrt[3]{2x-3}\ x\,dx$

$=\int \dfrac{u(u^3+3)}{2}\bullet\dfrac{3u^2}{2}du$

$=\dfrac{3}{4}\left(\int u^3(u^3+3)\right)du$

$=\dfrac{3}{4}\left(\int u^6du+\int 3u^3du\right)$

$=\dfrac{3}{4}\left(\dfrac{u^7}{7}+\dfrac{3u^4}{4}\right)+C=\dfrac{3u^7}{28}+\dfrac{9u^4}{16}+C$

Step 3: We change back to the variable $x$. $(u=\sqrt[3]{2x-3})$

$=\dfrac{3(\sqrt[3]{2x-3})^7}{28}+\dfrac{9(\sqrt[3]{2x-3})^4}{16}+C$

$=\dfrac{3(2x-3)^{\frac{7}{3}}}{28}+\dfrac{9(2x-3)^{\frac{4}{3}}}{16}+C$

$=(2x-3)^{\frac{4}{3}}[\dfrac{3(2x-3)}{28}+\dfrac{9}{16}]+C$

$=(2x-3)^{\frac{4}{3}}[\dfrac{6x-9}{28}+\dfrac{9}{16}]+C$

$=(2x-3)^{\frac{4}{3}}\dfrac{4(6x-9)+7(9)}{112}+C$

$=(2x-3)^{\frac{4}{3}}[\dfrac{24x+27}{112}]+C$

$=\dfrac{3(8x+9)}{112}(2x-3)^{\frac{4}{3}}+C$

$=\frac{3}{112}(8x+9)(2x-3)^{\frac{4}{3}}+C$

$=\frac{3}{112}(8x+9)(2x-3)(\sqrt[3]{2x-3})+C$

$=\frac{3}{112}(16x^2-6x-27)(\sqrt[3]{2x-3})+C$

**Method 2:** Use simple $u$-substitution without rationalizing the integrand.

Step 1: Let $u=2x-3$.

Then $\dfrac{du}{dx}=2$ and $dx=\dfrac{du}{2}$

Step 2: Substitute $u$ for $2x-3$, $\dfrac{du}{2}$ for $dx$ in $\int \sqrt[3]{2x-3}\ x\,dx$ to obtain $\int u^{\frac{1}{3}}x\bullet\dfrac{du}{2}$. We will express $x$ in terms of $u$. From $u=2x-3$. $x=\dfrac{u+3}{2}$. Substituting for $x$ in

$=\int u^{\frac{1}{3}}x\bullet\dfrac{du}{2}$, we obtain

$=\int \dfrac{u^{\frac{1}{3}}(u+3)}{2}\bullet\dfrac{du}{2}$

$=\dfrac{1}{4}\int u^{\frac{1}{3}}(u+3)du$

$=\dfrac{1}{4}\int\left(u^{\frac{4}{3}}+3u^{\frac{1}{3}}\right)du$

$=\dfrac{1}{4}(\dfrac{3u^{\frac{7}{3}}}{7}+\dfrac{9u^{\frac{4}{3}}}{4})+C$

$=\dfrac{3}{4}(\dfrac{u^{\frac{7}{3}}}{7}+\dfrac{3u^{\frac{4}{3}}}{4})+C$

$=\dfrac{3}{4}(\dfrac{4u^{\frac{7}{3}}+21u^{\frac{4}{3}}}{28})+C$

$=\dfrac{3}{112}(4u^{\frac{7}{3}}+21u^{\frac{4}{3}})+C$

$=\dfrac{3}{112}[4u^2u^{\frac{1}{3}}+21uu^{\frac{1}{3}}]+C$

$=\dfrac{3u^{\frac{1}{3}}}{112}[4u^2+21u]+C$

$=\frac{3}{112}\sqrt[3]{2x-3}\left[4(2x-3)^2+21(2x-3)\right]+C$

$=\frac{3}{112}\sqrt[3]{2x-3}\left[4(4x^2-12x+9)+42x-63\right]+C$

$=\frac{3}{112}\sqrt[3]{2x-3}\left[16x^2-48x+36+42x-63\right]+C$

$=\dfrac{3}{112}\sqrt[3]{2x-3}\left[16x^2-6x-27\right]+C$.

$=\dfrac{3}{112}\left[16x^2-6x-27\right]\sqrt[3]{2x-3}+C$