Example 3 Find $\lim_{x \to 0} \frac{x}{|x|}$.

Solution: $\lim_{x \to 0^+} f(x) = 1$**;** $\lim_{x \to 0^-} f(x) = -1$ and therefore the left and right limits are **not** equal, and the limit does not exist. (Figure below).

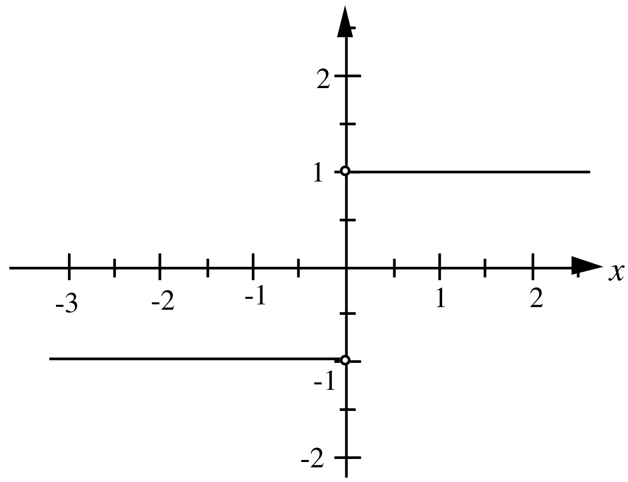

Figure: $f(x) = \frac{x}{|x|}$

For the left limit, choose say

$x = -0.00001$

$$\frac{x}{|x|} = \frac{-0.00001}{|-0.00001|} = \frac{-0.00001}{0.00001} = -1$$

For the right limit, choose say

$x = 0.00001$

$$\frac{x}{|x|} = \frac{0.00001}{|0.00001|} = \frac{0.00001}{0.00001} = 1$$

Clarification of the application of one-sided limits

1. The left limit, $\lim_{x \to a-} f(x) = \text{L}$ (exists: real and defined), and the right limit $\lim_{x \to a+} f(x) = \text{L}$ (exists: real and defined), and therefore, the left limit equals the right limit, and hence the normal limit, $\lim_{x \to a} f(x) = \text{L}$. (exists)

2. The left limit, $\lim_{x \to a-} f(x) = L_1$ exists (real and defined), and the right limit $\lim_{x \to a+} f(x) = L_2$ (exists (real and defined), but $L_1 \neq L_2$ and hence the normal limit, $\lim_{x \to a} f(x)$ does **not** exist.

3. Either $\lim_{x \to a-} f(x)$ exists and f is defined from the left , but $\lim_{x \to a+} f(x)$ does not exist, and in this case, the normal limit $\lim_{x \to a} f(x) = \lim_{x \to a-} f(x)$; or $\lim_{x \to a+} f(x)$ exists and f is defined from the right, but $\lim_{x \to a-} f(x)$ does not exist, and in this case, the normal limit $\lim_{x \to a} f(x) = \lim_{x \to a+} f(x)$.

Note that for $\lim_{x \to a} f(x)$ to exist, the function does **not** have to be defined at a.