Approach 2 Find $\int_0^3 \frac{5x}{\left(x^2+1\right)^3}dx$

Step 1: Let $u = x^2 + 1$

Then $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$

Step 2: Express the limits of integration in terms of u.

When $x = 0$, $u = 0^2 + 1 = 1$;

when $x = 3$, $u = 3^2 + 1 = 10$

Therefore, the limits in terms of u are $u = 1$ and $u = 10$

Step 3: In $\int_0^3 \frac{5x}{\left(x^2+1\right)^3}dx$,

replace 0 by 1; 3 by 10; $x^2 + 1$ by u; dx by $\frac{du}{2x}$. or $du = 2x\,dx$,

Then $\int_0^3 \frac{5x}{(x^2+1)^3}\,dx = \int_1^{10} \frac{5x}{u^3}\cdot\frac{du}{2x}$:

$= \int_1^{10} \frac{5}{u^3}\cdot\frac{du}{2}$

Step 4:

$$= \frac{5}{2}\int_1^{10}\frac{du}{u^3}$$

$$= \frac{5}{2}\int_1^{10}u^{-3}du$$

$$= \frac{5}{2}\left[\frac{u^{-3+1}}{-3+1}\right]_1^{10}$$

$$= \frac{5}{2}\left[\frac{u^{-2}}{-2}\right]_1^{10}$$

$$= -\frac{5}{4}\left[\frac{1}{u^2}\right]_1^{10}$$

$$= -\frac{5}{4}\left(\frac{1}{10^2} - \frac{1}{1^2}\right)$$

$$= -\frac{5}{4}\left(\frac{1}{100} - \frac{1}{1}\right)$$

$$= -\frac{5}{4}\left(\frac{-99}{100}\right) = \frac{99}{80}$$

Example 5 Find $\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x\,dx$ (Using Approach 2)

Step 1: Let $u = \sin 2x$

Then $\frac{du}{dx} = 2\cos 2x$ and

$dx = \frac{du}{2\cos 2x}$ or

$du = 2\cos 2x\,dx$

Step 2: Express the limits of integration in terms of u.

When $x = 0$, $u = \sin 0 = 0$;

when $x = \frac{\pi}{4}$,

$u = \sin 2 \bullet \frac{\pi}{4} = \sin\frac{\pi}{2} = 1$

Therefore, the limits in terms of u are $u = 0$ and $u = 1$.

Step 3: In $\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x\,dx$,

replace $\frac{\pi}{4}$ by 1; $\sin 2x$ by u;

dx by $\frac{du}{2\cos 2x}$ .or $du = 2\cos 2x\,dx$

to obtain

Step 4: $\int_0^1 u^3 \cos 2x \frac{du}{2\cos 2x}$

$$= \frac{1}{2}\int_0^1 u^3 du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1^4}{4} - \frac{0^4}{4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$$