Example 5 Show that $\int \csc x dx = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C$ (See p.298 for math similarity)

Step 1: $\int \csc x dx = \int \frac{1}{\sin x} dx$

$= \int \frac{1}{\sin x} \bullet \frac{\sin x}{\sin x} dx$

$= \int \frac{\sin x}{\sin^2 x} dx$

$= \int \frac{\sin x}{1-\cos^2 x} dx$

$(u = \cos x)$ $= \int \frac{\sin x}{1-u^2} \bullet \left(-\frac{du}{\sin x}\right)$

$(dx = -\frac{du}{\sin x})$ $= -\int \frac{1}{1-u^2} du$

$= -\int \frac{1}{(1+u)(1-u)} du$

$= -\left(\frac{1}{2}\int \frac{1}{1+u} du + \frac{1}{2}\int \frac{1}{1-u} du\right)$

(using partial fraction decomposition)

Step 2: $= -\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C$

$= \ln\left|\frac{1+u}{1-u}\right|^{-\frac{1}{2}} + C$

$= \ln\left|\frac{1+\cos x}{1-\cos x}\right|^{-\frac{1}{2}} + C$

$= \ln\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}} + C$

$\int \csc x dx = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C.$

Example 6 Show also that $\int \csc x dx = \ln|\csc x - \cot x| + C$

$\int \csc x dx = \ln\sqrt{\frac{(1-\cos x)}{(1+\cos x)} \bullet \frac{(1-\cos x)}{(1-\cos x)}} + C$ (also see p. 238)

$= \ln\sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} + C$

$= \ln\sqrt{\frac{(1-\cos x)^2}{\sin^2 x}} + C$

$= \ln\left|\frac{1-\cos x}{\sin x}\right| + C$

$= \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + C$

$\int \csc x dx = \ln|\csc x - \cot x| + C$ $\left(\frac{1}{\sin x} = \csc x; \frac{\cos x}{\sin x} = \cot x\right)$

$\therefore \int \csc x dx = \ln|\csc x - \cot x| + C = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C$

Lesson 42 Exercises

1. Find $\int \frac{dx}{\cos x - \sin x + 1}$; **2.** $\int \frac{1}{5+4\cos x} dx$; **3.** $\int \frac{1}{\sin x + 1} dx$

4. Show that $\int \csc x dx = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C.$

Answers **1.** $-\ln\left|1 - \tan\frac{x}{2}\right|$; **2.** $= \frac{2}{3}\text{Tan}^{-1}\left(3\tan\frac{x}{2}\right) + C$; **3.** $-\frac{2}{\tan\frac{x}{2}+1} + C.$