

**Example 4** Find  $\int \sec^5 x \, dx$

$$\begin{aligned} \text{Step 1: } \int \sec^5 x \, dx &= \int (\sec^2 x)^2 \sec x \, dx \\ &= \int (\tan^2 x + 1)^2 \sec x \, dx \\ &= \int (\tan^4 x + 2 \tan^2 x + 1) \sec x \, dx \\ &= \int \tan^4 x \sec x \, dx + 2 \int \tan^2 x \sec x \, dx + \int \sec x \, dx \end{aligned}$$

Step 2: Find the first two integrals using integration-by-parts.

$$\begin{aligned} &= \int \tan^3 x \cdot \tan x \sec x \, dx + 2 \int \tan x \cdot \tan x \sec x \, dx + \int \sec x \, dx \\ &\quad (\mathbf{1. } u = \tan^3 x; \, dv = \tan x \sec x \, dx; \, \mathbf{2. } u = \tan x \, dv = \tan x \sec x \, dx) \\ &= \tan^3 x \sec x - 3 \int \tan^2 x \sec^3 x \, dx + 2 \tan x \sec x - 2 \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

Step 3: Let  $\tan^2 x = \sec^2 x - 1$  in the first integral..

$$\begin{aligned} &= \tan^3 x \sec x - 3 \int (\sec^2 x - 1) \sec^3 x \, dx + 2 \tan x \sec x - 2 \int \sec^3 x \, dx + \int \sec x \, dx \\ &= \tan^3 x \sec x - 3 \int \sec^5 x + 3 \int \sec^3 x \, dx + 2 \tan x \sec x - 2 \int \sec^3 x \, dx + \int \sec x \, dx \\ \int \sec^5 x \, dx &= \tan^3 x \sec x - 3 \int \sec^5 x \, dx + \int \sec^3 x \, dx + 2 \tan x \sec x + \int \sec x \, dx \\ &\quad (\text{adding the } \int \sec^3 x \, dx \text{ terms}) \end{aligned}$$

Step 4: Add  $3 \int \sec^5 x \, dx$  to both sides of the equation; and

$$4 \int \sec^5 x \, dx = \tan^3 x \sec x + \int \sec^3 x \, dx + 2 \tan x \sec x + \int \sec x \, dx \quad (\text{Type 3 Int.})$$

Step 5: Obtain  $\int \sec^3 x \, dx$  from Example 2, Case 6 and find  $\int \sec x \, dx$ .

$$4 \int \sec^5 x = \tan^3 x \sec x + \frac{1}{2} \ln|\sec x + \tan x| + \frac{1}{2} \tan x \sec x + 2 \tan x \sec x + \ln|\sec x + \tan x|$$

$$4 \int \sec^5 x = \tan^3 x \sec x + \frac{3}{2} \ln|\sec x + \tan x| + \frac{5}{2} \tan x \sec x$$

$$\int \sec^5 x = \frac{\tan^3 x \sec x}{4} + \frac{3}{8} \ln|\sec x + \tan x| + \frac{5}{8} \tan x \sec x$$

$$\int \sec^5 x = \frac{\tan^3 x \sec x}{4} + \frac{5}{8} \tan x \sec x + \frac{3}{8} \ln|\sec x + \tan x| + C$$

**Example 5** Find  $\int \sec^6 x \, dx$

$$\begin{aligned} &\int \sec^6 x \, dx \\ &= \int (\sec^4 x)(\sec^2 x) \, dx \\ &= \int (\sec^2 x)^2 (\sec^2 x) \, dx \\ &= \int (\tan^2 x + 1)^2 (\sec^2 x) \, dx \quad (\sec^2 x = \tan^2 x + 1) \end{aligned}$$

$$= \int (u^2 + 1)(u^2 + 1) \, dx \quad (u = \tan x, \frac{du}{dx} = \sec^2 x, \, dx = \frac{du}{\sec^2 x}, \text{ or } du = \sec^2 x \, dx)$$

$$= \int (u^4 + 2u^2 + 1) \, du$$

$$= \frac{u^5}{5} + \frac{2u^3}{3} + u + C$$

$$= \frac{\tan^5 x}{5} + \frac{2 \tan^3 x}{3} + \tan x + C.$$