CHAPTER 37

From Trigonometric Functions to
Inverse Trigonometric Functions

("Exchange is no Robbery")

Lesson 96

Introduction; Definitions; Operations Involving Inverse Trigonometric Functions

We use inverse trigonometric functions in solving trigonometric equations. In calculus, we use inverse trigonometric functions to obtain formulas for various results involving algebraic functions.

In Chapter 31, we learned how to find the functional values (y-values), given the measures of the angles. There, we obtained the functional values using trigonometric tables, the electronic calculator or from memory. We also learned how to find the measures of angles given the functional values.
With inverse trigonometric functions, we are also given the functional value, say y, and we are asked to find an angle or a number, say x. The process of finding y and the process of finding x are inverse operations (in much the same way as multiplication and division are inverse operations).

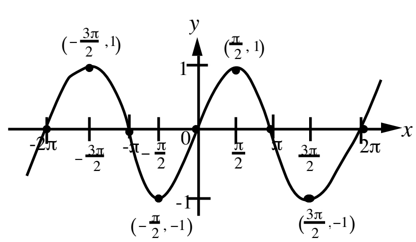
In Chapter 6, we also learned how to determine the inverses of algebraic functions, given the various functional forms. If the function was specified by a set form, we obtained its inverse by interchanging the first and second components of each ordered pair. With trigonometric functions, interchanging the first and second components of each ordered pair of numbers generally produce relations which are **not** functions because of the periodicity of the trigonometric functions.
Consider the **trigonometric relation** $y = \sin x$, which is specified by the set of ordered pairs,

$A = \{(-2\pi, 0), (-\frac{3\pi}{2}, 1), (-\pi, 0), (-\frac{\pi}{2}, -1), (0, 0), (\frac{\pi}{2}, 1), (\pi, 0), (\frac{3\pi}{2}, -1), (2\pi, 0)\}$.

This relation is also a **function** since no two distinct ordered pairs have the same first component.
(A function is a set of ordered pairs of numbers in which no two distinct ordered pairs have the same first component).
On interchanging the first and second components of each ordered pair, we obtain the inverse relation,

$B = \{(0, -2\pi), (1, -\frac{3\pi}{2}), (0, -\pi), (-1, -\frac{\pi}{2}), (0, 0), (1, \frac{\pi}{2}), (0, \pi), (-1, \frac{3\pi}{2})\ (0, 2\pi)\}$.

This new **relation** is **not a function** since the first, the third, the fifth, the seventh and the ninth ordered pairs have the same first component, namely 0, or also, the second, and the sixth ordered pairs have the same first component, namely 1. We observe that the original set, A, does **not** represent a **one-to-one function** and therefore, the inverse relation is not a function. The graph in the figure below fails the horizontal line test.



To make the set B a function, we can eliminate some of the ordered pairs so that no two distinct ordered pairs have the same first component. If we consider the ordered pair $(0,0)$ and choose ordered pairs to its right and ordered pairs to its left so that no two distinct ordered pairs have the same first component, we obtain the following: $C = \{(-1,-\frac{\pi}{2}),(0,0),(1,\frac{\pi}{2})\}$..This set C is a **function** since all the first components are different from one another. We observe also that in the original function given by set A, if we keep the fourth, the fifth and the sixth ordered pairs and drop the other pairs, we obtain $\{(-\frac{\pi}{2},-1),(0,0),(\frac{\pi}{2},1)\}$ which is a one-to-one function and therefore, its inverse relation is also a function.

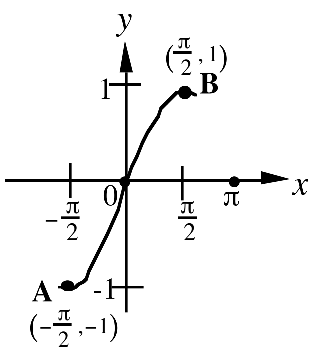


Fig 1. $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$-1 \le y \le 1$

Principal branch

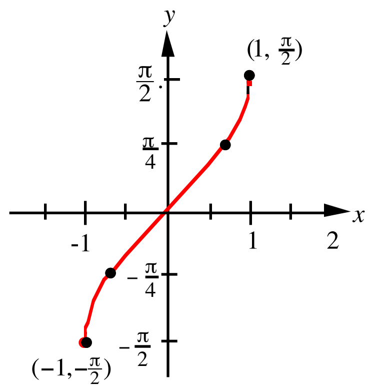


Fig. 2 $x = \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
$-1 \le x \le 1$

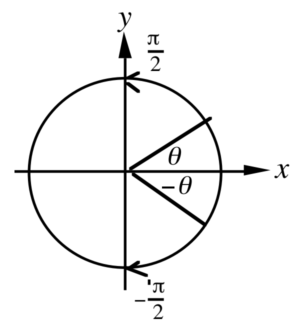


Fig. 3: For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

From above, we notice that the inverse relation for the sine function, $y = \sin x$, is generally **not** a function. Similarly, the inverse relations for $y = \cos x$, $y = \sec x$, $y = \csc x$, $y = \tan x$, and $y = \tan x$ are generally **not** functions.
Some of these inverse relations are many-valued and consist of a collection of single-valued functions known as branches. Since it is more convenient to work with functions, we will make these inverse relations become functions by restricting their ranges. By placing restrictions on the domains of those original functions which are **not** one-to-one, we obtain a particular branch called the **principal branch** of each inverse relation**.** Each branch **chosen** is an inverse function and we capitalize the first character of the functional name or its abbreviation. (Note that some authors do not adhere to this **capitalization** and in which case the principal branch and its range are to be understood).

Inverse Sine Function

Definition Given, the equation, $y = \sin x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, what is its inverse?

Since the inverse is found by **interchanging** the roles of x and y,

Domain ↓ Range ↓

the inverse of $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad -1 \le y \le 1$ is

$x = \sin y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad -1 \le x \le 1$

Range Domain

(Observe that the domain and range are interchanged.)

If we solve $x = \sin y$ for y, and use the notation, $\text{Sin}^{-1}x$, we obtain

$y = \text{Sin}^{-1}x$ (Recall previously that the inverse of $f(x)$ was symbolized $f^{-1}(x)$)

Another notation for this inverse is $y = \text{Arcsin } x$. We may use either $y = \text{Sin}^{-1}x$ or $y = \text{Arcsin } x$**.**

Example 1 If $\frac{\pi}{6} = \text{Arcsin}\left(\frac{1}{2}\right)$ or $\text{Sin}^{-1}\left(\frac{1}{2}\right)$, then $\sin\frac{\pi}{6} = \frac{1}{2}$	(is read " if $\frac{\pi}{6}$ is the angle or number whose sine is $\frac{1}{2}$.") (Memorize this example to help remember the definitions) $y = \text{Arcsin}\ x$ (y is the angle or the number whose sine is x.

Inverse Cosine Function

Definition Given, the equation, $y = \cos x \quad 0 \le x \le \pi$, what is its inverse?

The inverse is found by interchanging the roles of x and y.

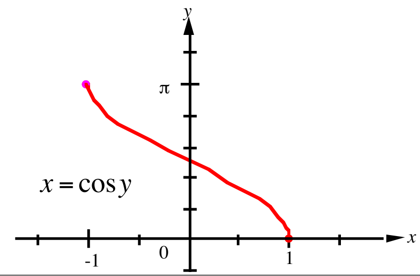
Domain ↓ Range ↓

The inverse of $y = \cos x$ for $0 \le x \le \pi, \ -1 \le x \le 1$ is

$x = \cos y$ for $0 \le y \le \pi, \ -1 \le x \le 1$

Range Domain

If we solve $x = \cos y$ for y, and use the notation, $\text{Cos}^{-1}x$, we obtain $y = \text{Cos}^{-1}x$ or $y = \text{Arccos}\ x$



Inverse Tangent Function

Definition Given, the equation, $y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, what is its inverse?

The inverse is found by interchanging the roles of x and y.

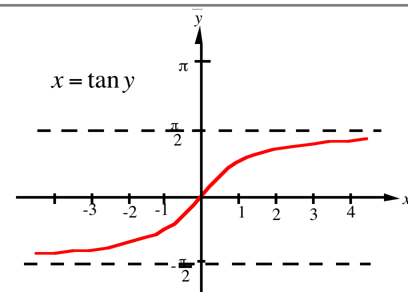
Domain ↓ Range ↓

The inverse of $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}, \ -\infty < y < \infty$ is

$x = \tan y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}, \ -\infty < x < \infty$

Range Domain

If we solve $x = \tan y$ for y, and use the notation, $\text{Tan}^{-1}x$, we obtain $y = \text{Tan}^{-1}x$ or $y = \text{Arctan}\ x$



Other Inverse Trigonometric Functions

The inverse functions $\text{Csc}^{-1}x$, $\text{Sec}^{-1}x$, and $\text{Cot}^{-1}x$ can be expressed in terms of $\text{Sin}^{-1}x$, $\text{Cos}^{-1}x$, and $\text{Tan}^{-1}x$ respectively as follows:

$\text{Csc}^{-1}x = \text{Sin}^{-1}\left(\frac{1}{x}\right)$ for $|x| \ge 1$; $\text{Sec}^{-1}x = \text{Cos}^{-1}\left(\frac{1}{x}\right)$ for $|x| \ge 1$; and $\text{Cot}^{-1}x = \text{Tan}^{-1}\left(\frac{1}{x}\right)$.

For example, $\text{Csc}^{-1}2 = \text{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. That is, to find $\text{Csc}^{-1}2$, first find the reciprocal of 2, which is $\frac{1}{2}$ and then find $\text{Sin}^{-1}\left(\frac{1}{2}\right)$, which is $\frac{\pi}{6}$. This approach gives only the reference angle and you have to place the angle in standard position for the original reciprocal inverse function according to the quadrant for its definition. For example, if x is negative and we use $\text{Cot}^{-1}x = \text{Tan}^{-1}\left(\frac{1}{x}\right)$, we must place the reference angle in the second quadrant for $\text{Cot}^{-1}x$, and write the standard position angle.

Choice of the ranges and quadrants for the inverse functions (See also p. 629)

If x is positive, $\text{Sin}^{-1}x$ $\text{Cos}^{-1}x$, $\text{Tan}^{-1}x$, $\text{Csc}^{-1}x$, $\text{Sec}^{-1}x$, and $\text{Cot}^{-1}x$ are all in the first quadrant, or appropriately quadrantal. However, if x is negative, $\text{Sin}^{-1}x$,$\text{Csc}^{-1}x$, or $\text{Tan}^{-1}x$ are in the fourth quadrant or appropriately quadrantal and they are expressed as negative angles; while if x is negative, $\text{Cos}^{-1}x$, $\text{Sec}^{-1}x$, and $\text{Cot}^{-1}x$ are in the second quadrant or appropriately quadrantal.

Note the difference between the quadrants for, say, the angle, $\text{Sin}^{-1}x$ (Fig. 3, previous page) and the quadrants for the graph of $y = \text{Sin}^{-1}x$ (Fig. 2, previous page)..The graph consists of the points (x, y).

We summarize the definitions of the **inverse trigonometric functions** below. The restrictions made to obtain the inverse functions are arbitrary, and other branches may be used. The above branches are the ones that are usually used.

Function	Domain	Range	Inverse	Notation	Domain	Range
$y = \sin x$,	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,	$-1 \le y \le 1$	$x = \sin y$,	$y = \text{Arc}\sin x$ or $\text{Sin}^{-1}x$;	$-1 \le x \le 1$;	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos x$,	$0 \le x \le \pi$,	$-1 \le y \le 1$	$x = \cos y$,	$y = \text{Arc}\cos x$ or $\text{Cos}^{-1}x$;	$-1 \le x \le 1$;	$0 \le y \le \pi$
$y = \tan x$,	$-\frac{\pi}{2} < x < \frac{\pi}{2}$,	$-\infty < y < \infty$	$x = \tan y$,	$y = \text{Arc}\tan x$ or $\text{Tan}^{-1}x$;	$-\infty < x < \infty$;	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot x$,	$0 < x < \pi$,	$-\infty < y < \infty$	$x = \cot y$,	$y = \text{Arc}\cot x$ or $\text{Cot}^{-1}x$;	$-\infty < x < \infty$;	$0 < y < \pi$
(Also: $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \frac{\pi}{2}$)				or $y = \text{Cot}^{-1}x = \text{Tan}^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \text{Tan}^{-1}x$;		$0 < y < \pi$
$y = \sec x$,	$0 \le x \le \pi$, $x \ne \frac{\pi}{2}$,	$\lvert y \rvert \ge 1$	$x = \sec y$,	$y = \text{Arc}\sec x$ or $\text{Sec}^{-1}x$;	$\lvert x \rvert \ge 1$,	$0 \le y \le \pi$, $y \ne \frac{\pi}{2}$
			Also,	$y = \text{Sec}^{-1}x = \text{Cos}^{-1}\left(\frac{1}{x}\right)$;	$\lvert x \rvert \ge 1$	$0 \le y \le \pi$, $y \ne \frac{\pi}{2}$
$y = \csc x$,	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x \ne 0$,	$\lvert y \rvert \ge 1$	$x = \csc y$,	$y = \text{Arc}\csc x$ or Csc^{-1};	$\lvert x \rvert \ge 1$,	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(Also: $\text{Sec}^{-1}x + \text{Csc}^{-1}x = \frac{\pi}{2}$)			Also,	$y = \text{Csc}^{-1}x = \text{Sin}^{-1}\left(\frac{1}{x}\right)$;	$\lvert x \rvert \ge 1$,	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$

Inverse Operations

Finding the **sine** of an angle or a number and finding the inverse sine or **arcsine** are inverse processes of each other in much the same way that **multiplication and division** are inverse operations of each other. Two operations which are inverses of each other undo (reverse the action of) each other. Thus the function $\text{Arcsin}\, x$ reverses the action of $\sin x$, and vice versa.

Example 2 Find Arcsin (0.866). (Note that Arcsin (.866) is an angle or a number.)

Step 1: Let $y = \text{Arcsin}\,(0.866)$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (We will find y, the angle whose sine is .866)

Two conditions must be satisfied: 1. $\sin y = 0.866)$; **2**. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Step 2: Find the reference angle for the angle whose sine 0.866

From memory or calculator (or tables), the reference angle is $\frac{\pi}{3}$. (Note that $\frac{\pi}{3} = 60°$)

Step 3: Find the quadrant in which the sine function is positive.
The sine function is positive in the **first** and second quadrants.

Since by definition, Arcsin x must be in either the **first** or the fourth quadrant, ($-\frac{\pi}{2} \le y \le \frac{\pi}{2}$),

Arcsin x is in the **first** quadrant where it is a positive angle.

Therefore, Arcsin $(0.866) = \frac{\pi}{3}$. (Note that $-\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$)

Example 3 Find Arcsin (-0.5). (Note that Arcsin (- 0.5) is an angle or a number.)

Step 1: Let $y = \text{Arcsin}\,(-0.5)$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (We will find y, the angle whose sine is -0.5)

Step 2: Find the reference angle for the angle whose sine is - 0.5.

From tables or memory (or calculator), the reference angle is $\frac{\pi}{6}$.

Step 3: Find the quadrant in which the sine function is negative. The sine function is negative in the third and **fourth** quadrants. Since Arcsin x must be in either the first or the **fourth** quadrant,

($-\frac{\pi}{2} \le y \le \frac{\pi}{2}$), Arcsin x is in the fourth quadrant where it is a negative angle.

Therefore, Arcsin $(-0.5) = -\frac{\pi}{6}$. Summarily, $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1}x$

Example 4 Find $\text{Cot}^{-1}(-2.7)$ using a calculator.

Method 1: Step 1: Find $\frac{1}{2.7} = 0.37037037$ (Summarily, $\text{Cot}^{-1}(-2.7) = \pi - \text{Cot}^{-1}(2.7) = \pi - \text{Tan}^{-1}(\frac{1}{2.7})$)

Step 2: Find $\text{Tan}^{-1}\left(\frac{1}{2.7}\right) = \text{Tan}^{-1}(0.37037037) = 0.35$ <--- Reference angle in radians

The reference angle for $\text{Cot}^{-1}(-2.7) = 0.35$. By definition, when x is negative, $\text{Cot}^{-1}x$ is in the second quadrant and the terminal side of the angle is in the second quadrant.

Therefore, $\text{Cot}^{-1}(-2.7) = \pi - 0.35 \boxed{= 2.79}$.

Method 2: Applying $\text{Cot}^{-1}x + \text{Tan}^{-1}x = \frac{\pi}{2}$ and $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1}x$, $-\infty < x < \infty$, see also p.629.

$$\text{Cot}^{-1}(-2.7) = \frac{\pi}{2} - \text{Tan}^{-1}(-2.7) = \frac{\pi}{2} - [-\text{Tan}^{-1}(2.7)] \boxed{= \frac{\pi}{2} + \text{Tan}^{-1}(2.7)} = 1.57 + 1.216 \boxed{= 2.79}$$

General Composition of Trigonometric and Inverse Trigonometric Functions

Consider the following question at the beginning level of trigonometry.

" If $\sin x = \frac{1}{2}$, and the terminal side of x is in the first quadrant, find $\cot x$ ".

However, at the present level, we might pose the same question as:

" find $\cot\left(\text{Sin}^{-1}\frac{1}{2}\right)$ " <--Find the cotangent of the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$.

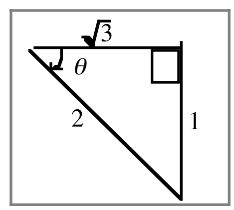
Example 5 Find the value of $\cot\left(\text{Sin}^{-1}\left(\frac{1}{2}\right)\right)$.

Solution: Same as if $\sin\theta = \frac{1}{2}$, and θ is in the first quadrant, find $\cot\theta$.

(or If $\sin\theta = \frac{1}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, find $\cot\theta$.) (Note $\sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$, $\theta = \frac{\pi}{6}$)

Step 1: Draw a right triangle, and let $\text{Sin}^{-1}\left(\frac{1}{2}\right) = \theta$

Then $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{1}{2}$, so that the side opposite θ is 1, and the hypotenuse is 2.



Step 2 Using the Pythagorean theorem, the side adjacent to θ is found to be $\sqrt{3}$.

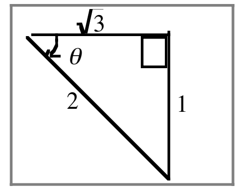
Step 3: Since $\sin\theta = +\frac{1}{2}$, $\text{Sin}^{-1}\left(\frac{1}{2}\right)$ is in the first quadrant

Step 4: $\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{\sqrt{3}}{1} = \sqrt{3}$

.Step 5: Determine the sign of $\cot\theta$. Since $\text{Sin}^{-1}\left(\frac{1}{2}\right)$ is in the first quadrant, where

$\cot\theta$ is also positive, $\cot\left(\text{Sin}^{-1}\left(\frac{1}{2}\right)\right) = \sqrt{3}$.

Example 6 Find the value of $\cot\left(\text{Sin}^{-1}\left(-\frac{1}{2}\right)\right)$. Same as if $\sin\theta = -\frac{1}{2}$, and θ is in the fourth quadrant, find $\cot\theta$.(or If $\sin\theta = -\frac{1}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, find $\cot\theta$) See also p.629



Step 1: Draw a right triangle, and let $\text{Sin}^{-1}\left(\frac{1}{2}\right) = \theta$. Then $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{1}{2}$, so that the side opposite θ is 1, and the hypotenuse is 2.

Step 2 Using the Pythagorean theorem, the side adjacent to θ is found to be $\sqrt{3}$

Step 3: $\sin\theta = -\frac{1}{2}$ (negative), $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$ is in the fourth quadrant, See also p.629

Step 4: $\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{\sqrt{3}}{1} = \sqrt{3}$.

Step 5: Determine the sign of $\cot\theta$. Since $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$ is in the fourth quadrant, where the cotangent is negative, $\cot\left(\text{Sin}^{-1}\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$.

Special Cases of Composition of Trigonometric Functions

Example The function $y = \text{Sin}^{-1}x$ and the restricted sine function $y = \sin x$ are inverses of each other, and as such, each reverses the action of the other,

Definitions

$\text{Sin}^{-1}(\sin x) = x;\quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ <-domain of the inside function $\sin x$	Note
$\sin(\text{Sin}^{-1}x) = x;\quad -1 \le x \le 1$ <-domain of the inside function $\sin^{-1}x$	$\pi \approx 3.14$
$\cos(\text{Cos}^{-1}x) = x;\quad -1 \le x \le 1$ <-domain of the inside function $\cos^{-1}x$	$\frac{\pi}{2} \approx \frac{3.142}{2} = 1.57$
$\text{Cos}^{-1}(\cos x) = x;\quad 0 \le x \le \pi$ <-domain of the inside function $\cos x$	
$\tan(\text{Tan}^{-1}x) = x;\quad -\infty < x < \infty$ <-domain of the inside function $\tan^{-1}x$	
$\text{Tan}^{-1}(\tan x) = x;\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ <-domain of the inside function $\tan x$	

Examples:

Case 1

Example 1 Evaluate: $\text{Sin}^{-1}(\sin\frac{\pi}{6})$ (A)

Step 1: Evaluate the inside function. From memory or tables $\sin\frac{\pi}{6} = \frac{1}{2}$

Step 2: Evaluate $\text{Sin}^{-1}\left(\frac{1}{2}\right)$

Step 3: Find the angle in radians or number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$

From memory or tables the angle is $\frac{\pi}{6}$.

Therefore $\text{Sin}^{-1}(\sin\frac{\pi}{6}) = \frac{\pi}{6}$ (B)

Observe that we could have obtained the angle $\frac{\pi}{6}$ by inspection and applying the definition,

$$\sin^{-1}(\sin x) = x; \text{ if } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \text{ or } -1.57 \le x \le 1.57 \quad \text{(C)}$$

Here, $x = \frac{\pi}{6} \approx 0.52$ and $-\frac{\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}$ or $-1.57 \le 0.52 \le 1.57$

Example 2: Evaluate $\text{Sin}^{-1}(\sin\frac{\pi}{4})$

The argument $\frac{\pi}{4}$ of the inside function is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and satisfies the domain

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ of the restricted sine function.

Therefore, the domain of the function is satisfied, and definition (C) above is applicable, and

$\text{Sin}^{-1}(\sin\frac{\pi}{4}) = \frac{\pi}{4}$ (The inverse sine function undoes what the sine function does)

Case 2 Example 3 Evaluate $\sin\left(\mathrm{Sin}^{-1}\frac{1}{2}\right)$

The argument $\frac{1}{2}$ of the inside function is between -1 and 1, and satisfies the domain $-1 \le x \le 1$. Therefore, $\sin\left(\mathrm{Sin}^{-1}\frac{1}{2}\right) = \frac{1}{2}$. (The sine function undoes what the inverse sine function does.)

Example 4 Evaluate $\sin\left(\mathrm{Sin}^{-1}4\right)$

The argument 4 of the inside function does not satisfy the domain $-1 \le x \le 1$ of $\mathrm{Sin}^{-1}x$. Therefore, $\sin\left(\mathrm{Sin}^{-1}4\right)$ is not defined. There is no angle or number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 4.

Case 3 Example 5 Evaluate $\mathrm{Sin}^{-1}(\sin\frac{3\pi}{4})$

The argument $\frac{3\pi}{4}$ of the inside function is **not** between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ does not satisfy the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and therefore $\mathrm{Sin}^{-1}(\sin x) = x$ is not applicable. However, $\sin\frac{3\pi}{4} = 0.7071$

$\mathrm{Sin}^{-1}(0.7071) = 0.7854$ (Note: $\frac{3\pi}{4} \approx 2.3562$, and therefore, $\sin^{-1}(\sin\frac{3\pi}{4})$ is not equal to $= \frac{3\pi}{4}$)

Therefore, $\mathrm{Sin}^{-1}(\sin\frac{3\pi}{4}) = 0.7854$. **Clarification of the ranges of inverse trig functions follows:**

Range of Inverse Trig	Functions when $x \ge 0$	Range of Inverse Trig	Functions when $x < 0$
$0 \le \mathrm{Sin}^{-1}x \le \frac{\pi}{2}$ (Ist qd)	$0 < \mathrm{Cot}^{-1}x \le \frac{\pi}{2}$ (Ist qd)	$-\frac{\pi}{2} \le \mathrm{Sin}^{-1}x < 0$ (4th qd)	$\frac{\pi}{2} < \mathrm{Cot}^{-1}x < \pi$ (2nd qd)
$0 \le \mathrm{Cos}^{-1}x \le \frac{\pi}{2}$ (Ist qd)	$0 \le \mathrm{Sec}^{-1}x < \frac{\pi}{2}$ (Ist qd)	$\frac{\pi}{2} < \mathrm{Cos}^{-1}x \le \pi$ (2nd qd)	$\frac{\pi}{2} < \mathrm{Sec}^{-1}x \le \pi$ (2nd qd))
$0 \le \mathrm{Tan}^{-1}x < \frac{\pi}{2}$ (Ist qd)	$0 < \mathrm{Csc}^{-1}x \le \frac{\pi}{2}$ (Ist qd)	$-\frac{\pi}{2} < \mathrm{Tan}^{-1}x < 0$ (4th qd)	$-\frac{\pi}{2} \le \mathrm{Csc}^{-1}x < 0$ (4th qd)

qd = quadrant

Inverse Cofunction Identities

1. $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = \frac{\pi}{2}$, $[-1,1]$; **2.** $\mathrm{Tan}^{-1}x + \mathrm{Cot}^{-1}x = \frac{\pi}{2}$, $(-\infty,\infty)$; **3** $\mathrm{Sec}^{-1}x + \mathrm{Csc}^{-1}x = \frac{\pi}{2}$, $|x| \ge 1$

Each identity is subject to the domain of definition, Application: $\mathrm{Cot}^{-1}x = \frac{\pi}{2} - \mathrm{Tan}^{-1}x$

For proofs of these identities, see Appendix B, p.660.

Negative angle relationships

1. $\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x$; **3**. $\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1}x$; **5**. $\mathrm{Sec}^{-1}(-x) = \pi - \mathrm{Sec}^{-1}x$;

2. $\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x$; **4**. $\mathrm{Cot}^{-1}(-x) = \pi - \mathrm{Cot}^{-1}x$; **6**. $\mathrm{Csc}^{-1}(-x) = -\mathrm{Csc}^{-1}x$.

Lesson 96 Exercises

Find the following

1.. $\mathrm{Arc}\sin\frac{1}{2}$; **2**. $\mathrm{Arc}\tan\sqrt{3}$; **3**. $\mathrm{Arc}\cos\frac{1}{2}$

4. $\mathrm{Arc}\sin\left(-\frac{\sqrt{3}}{2}\right)$; **5**. $\mathrm{Sin}^{-1}1$; **6**. $\mathrm{Arc}\cot(-0.33)$

Without using tables or a calculator find :

7. $\cot\left(\mathrm{Arc}\sin\left(-\frac{1}{4}\right)\right)$; **8**. $\mathrm{Arc}\cos\left(\cos\left(\frac{15\pi}{11}\right)\right)$

11. Express $\tan\left(\mathrm{Arc}\sin x\right)$ in terms of x.

Is each of the following true for all x:? Why?

9. $\mathrm{Arc}\sin\left(\sin x\right) = x$; **10**. $\sin\left(\mathrm{Arc}\sin x\right) = x$

12. Express $\cos\left(\mathrm{Arc}\tan x\right)$ in terms of x.

13. True or false $\tan(\mathrm{Tan}^{-1}x) = x$ for all x.

Answers : **1**. $\frac{\pi}{6}$; **2**. $\frac{\pi}{3}$; **3**. $\frac{\pi}{3}$; **4**. $-\frac{\pi}{3}$; **5**. $\frac{\pi}{2}$; **6**. 1.88; **7**. $-\sqrt{15}$; **8**. $\frac{7\pi}{11} \approx 1.999$; **9**. No : $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$; **10**. No : $-1 \le x \le 1$; **11**. $\frac{x}{\sqrt{1-x^2}}$; **12**. $\frac{1}{\sqrt{x^2+1}}$; **13**. True. $-\infty < x < \infty$

Note: In Problems **11** and **12, t**he composition of a trigonometric function with an inverse trigonometric function is an algebraic expression.