Extra Topics
Appendix E & Appendix F
Complex Numbers and
Polar coordinate System

APPENDIX E
Complex Numbers

Lesson 41
Definitions; Basic Operations with Complex Numbers

We use complex numbers in the study of electricity and magnetism, in the analysis of feed-back systems especially by the root-locus method, Bode analysis and Nyquist analysis.

Definition, Powers of i, Square Root of Negative Numbers

Sometimes, in attempting to solve certain polynomial equations, we arrive at situations in which we have to find the square roots of negative numbers.

Example: If $x^2 = -1$, then $x = \pm\sqrt{-1}$
Since, for the set of real numbers, there is no provision for the square root of a negative number, we introduce a number "i" which we call the imaginary unit, with the following definition:

Definition: $i = \sqrt{-1}$ or $i^2 = -1$ (We will use both forms of the definition; memorize them)

For example, $(\sqrt{-1})(\sqrt{-1}) = (i)(i) = i^2 = -1$

Powers of i (Cyclical property of i or i^2) **:** All powers of i are either equal to ±1 or $\pm i$.
Examples

(a) $i^2 = -1$
(b) $i^3 = (i^2)i = -1(i) = -i$

(c) $i^4 = (i^2)(i^2) = (-1)(-1) = 1$
(d) $i^{10} = (i^4)(i^4)(i^2) = (1)(1)(-1) = -1$

Note above: Even powers of i are equal to ±1. Odd powers of i are equal to $\pm i$.

Note that the imaginary unit "i" is only a tool in mathematics, and that it is not more imaginary (in the literal sense) than the real number 3. The introduction of the imaginary unit allows us to find the square roots of negative numbers.

Example Find the square root: (a) -4 ; (b) -25 ; (c) Simplify: $\sqrt{-15}$.

Solution

(a) $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$; $= i(2)$; $\sqrt{-4} = 2i$

(*b*) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$; $= i(5)$; $\sqrt{-25} = 5i$.

(*c*) $\sqrt{-15} = (\sqrt{-1})(\sqrt{15})$; $= i\sqrt{15}$ or $\sqrt{15}i$

Note: In (c) we prefer the first form of the answer; because, sometimes, if we are not careful in writing "*i*", the "*i*" may look as if it is under the radical sign . However, if no radical is involved we shall leave the answers as in (a) and (b) above. In some old textbooks, you may find "*i*" written after the radical.

Generally, $\sqrt{-b} = (\sqrt{-1})(\sqrt{b})$ ($b \geq 0$)
$= i\sqrt{b}$

The introduction of the imaginary unit helps us to expand the real number system to a more general system called the complex number system.

If we denote a complex number by *z*, then $z = a + b$i, where *a* and *b* are real numbers; *a* is called the real part and *b* is called the imaginary part. If $b = 0$, we have a pure real number (e.g., $z = 3 + 0i = 3$). If $a = 0$, we have a pure imaginary number ($z = 0 + 2i = 2i$). Thus, the product bi is called a pure imaginary number.

Distinction Between Roots of Negative Numbers and Roots of Equations

Roots of numbers (Principal Roots)**:** $\sqrt{-b} = \text{i}(\sqrt{b})$ ($b \geq 0$)

Example (a) $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$
$= 2i$.

(b) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$
$= 5i$.

Roots of equations: Here, we must note that we have more than one root. (two roots.)

Example 1: Consider the solution to the equation $x^2 = -25$.

Solution $x^2 = -25$
$x = \pm\sqrt{-25}$
$= \pm 5i$, which means $x = +5i$ or $x = -5i$.

Example 2 Solve for *x*:: $x^2 = -4$

Solution

$x = \pm\sqrt{-4}$
$x = \pm 2i$ (or $x = +2i$ or $x = -2i$)

Addition and Subtraction of Complex Numbers

In adding complex numbers, we shall add the real parts, and then add the imaginary parts (in much the same way as we add like terms in polynomial addition).

Perform the indicated operations, leaving the answers in the form $a + bi$.

Example 1 Simplify: $(-3 + 5i) + (-2 +7i)$

Step 1: Remove the parentheses.

$$(-3 + 5i) + (-2 + 7i)$$

$$= -3 + 5i - 2 + 7i$$

Step 2: Add the real parts, and add the imaginary parts.

$$-3 + 5i - 2 + 7i$$
$$= - 5 + 12i \text{ .}$$

Scrapwork:
For the real parts: -3 - 2 = - 5
For the imaginary parts: 5+ 7 = 12

Example 2 Simplify: $(6 - 5i) - (-3 + 2i)$

Solution Remove the parentheses and add.

$$(6 - 5i) - (-3 + 2i)$$
$$= 6 - 5i + 3 - 2i$$

$$= 9 - 7i$$

Example 3 Simplify: $(5 + 2i) + (-3 - 6i)$
Solution
Remove the parentheses and add.
$(5 + 2i) + (-3 - 6i)$
$= 5 + 2i - 3 - 6i$
$= 5 - 3 + 2i - 6i$ <-------you may skip this step.
$= 2 - 4i$

or adding vertically:

$$\begin{array}{r} 5 + 2i \\ \underline{-3 - 6i} \\ 2 - 4i \end{array}$$ (Adding).

Example 4 Simplify : $(3 + 4i\} - (2 - 5i)$ (subtraction)
Solution
Remove the parentheses and add.
$(3 + 4i) - (2 - 5i)$
$= 3 + 4i - 2 + 5i$
$= 3 - 2 + 4i + 5i$ <-------you may skip this step.
$= 1 + 9i$

Example 5 Simplify : $(6 - 3i) + (2 + 3i)$

Solution

$(6 - 3i) + (2 + 3i)$
$= 6 - 3i + 2 + 3i$
$= 8 + 0i$
$= 8$

Multiplication of Complex Numbers

The approach here is multiply, replace i^2 by -1 (or higher powers of i by ± 1 or $\pm i$) , add the real parts and add the imaginary parts.

Example 1 Multiply $-4 - 2i$ and $-5 + i$

Solution

Step 1: Multiply as you multiply binomials

$(-4 - 2i)(-5 + i)$
$= -4(-5 + i) + (-2i)(-5 + i)$ <--- You may skip this step.
$= 20 - 4i + 10i - 2i^2$

Step 2: (Replace i^2 by -1, since $i^2 = -1$ by definition)

$= 20 - 4i + 10i - 2(-1)$
$= 20 + 6i + 2$

Step 3: (Add the like terms: add the real parts; and add the imaginary parts)

$= 22 + 6i$

Example 2 Multiply $3 + 2i$ and $4 + 5i$

Procedure: Multiply, replace i^2 by -1, and simplify.

$(3 + 2i)(4 + 5i)$
$= 12 + 15i + 8i + 10i^2$
$= 12 + 23i + 10(-1)$
$= 12 + 23i - 10$
$= 2 + 23i$ (Adding)

Note: All powers of i are either equal to ± 1 or ± i. For example $i^3 = -i$, $i^4 = 1$, $i^{10} = -1$

Example 3 Simplify : $4(6 - 3i)$

Solution

$4(6 - 3i)$
$= 24 - 12i$

Example 4 Simplify: $(4 + 2i)(3 - 6i)$

Solution The above implies multiplication.

Multiply, replace i^2 by -1, and add the like terms.

$(4 + 2i)(3 - 6i)$
$= 12 - 24i + 6i - 12i^2$
$= 12 - 24i + 6i - 12(-1)$
$= 12 - 24i + 6i + 12$
$= 24 - 18i$ (Note: $12 + 12 = 24$ and $-24i + 6i = -18i$)

Example 5 Simplify: $(7 + 2i)(7 - 2i)$

Solution

$(7 + 2i)(7 - 2i)$
$= 49 - 14i + 14i - 4i^2$
$= 49 - 4(-1)$
$= 49 + 4$
$= 53$

Multiplication of the square roots of negative numbers

Example 1 Find the product of $(\sqrt{-8})$ and $(\sqrt{-2})$

Solution

Step 1: Change to complex number forms.

$\sqrt{-8} = \sqrt{-1}(\sqrt{8})$
$= i\sqrt{8}$

Similarly, $\sqrt{-2} = i\sqrt{2}$

Step 2: Multiply the complex forms now.

$(\sqrt{-8})(\sqrt{-2}) = i\sqrt{8}\, i\sqrt{2}$
$= i^2\sqrt{16}$ $(\sqrt{16} = 4)$
$= i^2\ (4)$
$= (-1)(4)$ $(i^2 = -1)$
$= -4$

In the above problem (Example 1), you may skip Step 1 and show only Step 2.

Example 2 Find the product $(\sqrt{-49})(\sqrt{-25})$

Solution

Step 1: Change to complex number forms.

Then, $\sqrt{-49} = i\sqrt{49} = i(7) = 7i$

<---You may do Step 1 mentally and show only Step 2.

$\sqrt{-25} = i\sqrt{25} = i(5) = 5i$

Step 2: Multiply the complex forms now.

Then, $(\sqrt{-49})(\sqrt{-25}) = (7i)(5i)$
$= (7)(5)i^2$
$= 35(-1)$ $(i^2 = -1)$
$= -35$

We must note in the last two examples that it was necessary first to express the square roots of the negative numbers in terms of *i* before proceeding to multiply. Failure to do this may result in error such as the following:

Wrong procedure---> $(\sqrt{-8}(\sqrt{-2}) = \sqrt{(-2)(-8)}$
$= \sqrt{16}$
$= 4$, **which is a wrong answer**.

Generally, it is true that $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$ if *a* and *b* are positive but it is not true if *a* and *b* are negative.

Complex Conjugates

Definition : The **conjugate** of a given binomial is another binomial that differs from the given binomial only in the sign of one of the terms. The conjugate of $a + b$ is $a - b$; and the conjugate of $a - b$ is $a + b$.

The **conjugate** of $a + bi$ is $a - bi$. The conjugate of $a - bi$ is $a + bi$. A complex number and its conjugate differ only in the sign of the imaginary part. To find the conjugate of a complex number, change the sign of the imaginary part and keep the sign of the real part unchanged.

Examples: The conjugate of $2 + 3i$ is $2 - 3i$. The conjugate of $4 - 2i$ is $4 + 2i$.

The conjugate of $7 - 5i$ is $7 + 5i$; The conjugate of $-3 + 6i$ is $-3 - 6i$.

The conjugate of $-2 -3i$ is $-2 + 3i$.; The conjugate of $4i$ is $- 4i$.

The conjugate of 7 is 7

The product of a complex number and its conjugate is a real number.

The product of $-2 -3i$ and $-2 + 3i$. $=4 - 6i + 6i - 9i^2 = 4 - 9(-1) = 4 + 9 =$ **13**, a real number.

The product of $a + bi$ and $a - bi = a^2 + b^2$

Some uses of complex conjugates: We may use the complex conjugate of a complex number to rationalize the denominator of a fraction or to divide by a complex number.

Division of Complex Numbers

Example 1 Simplify : $\dfrac{2 + 3i}{4 - 5i}$ or divide $2 + 3i$ by $4 - 5i$

To simplify the above complex number is meant we are to write it in the form $z = a + bi$.
The operation here is similar to that of the rationalization of denominators of radical expressions.
To simplify the above expression, we **multiply both the denominator and the numerator by the conjugate of the denominator.**

Solution The conjugate of $4 - 5i$ is $\mathbf{4 + 5i}$. (See also the note below)
Multiply both the denominator and the numerator by $4 + 5i$.

Then, we obtain:

$$\frac{2 + 3i}{4 - 5i}$$
$$= \frac{(2 + 3i)}{(4 - 5i)} \cdot \frac{\mathbf{(4 + 5i)}}{\mathbf{(4 + 5i)}}$$
$$= \frac{8 + 10i + 12i + 15i^2}{16 + 20i - 20i - 25i^2}$$
$$= \frac{8 + 22i + 15(-1)}{16 + 0 - 25(-1)}$$
$$= \frac{8 + 22i - 15}{16 + 25}$$
$$= \frac{8 + 22i - 15}{41}$$
$$= \frac{-7 + 22i}{41}$$
$$= -\frac{7}{41} + \frac{22}{41}i$$

We may observe above that the **denominator** does **not** contain the imaginary unit "i", even though the numerator contains the imaginary unit.

Note above that we could have multiplied by $-4 - 5i$. (Try it.) It is therefore not critical in this problem which terms should differ in sign, (so far as the rationalization of the denominator is concerned) provided that either the real parts differ in sign or the imaginary parts differ in sign, but **not** both. We may produce a minus sign in the denominator which we can take care of as usual.

Example 2 Simplify: $\frac{5 + 4i}{-3 + 2i}$

Solution Multiply both the denominator and the numerator by -3 - 2i (The conjugate of -3 + 2i).

$$\frac{5 + 4i}{-3 + 2i}$$

$$= \frac{(5 + 4i)}{(-3 + 2i)} \cdot \frac{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}$$

$$= \frac{-15 - 10i - 12i - 8i^2}{9 + 6i - 6i - 4i^2}$$

$$= \frac{-15 - 10i - 12i - 8(-1)}{9 + 0 - 4(-1)}$$

$$= \frac{-15 - 22i + 8}{9 + 4}$$

$$= \frac{-7 - 22i}{13}$$

$$= -\frac{7}{13} - \frac{22}{13}i \qquad (-15 + 8 = -7)$$

Example 3 Simplify: $\frac{4 - 2i}{i}$

Solution

$$\frac{4 - 2i}{i}$$

$$= \frac{(4 - 2i)}{i} \frac{\boldsymbol{(-i)}}{\boldsymbol{(-i)}}$$ (The conjugate of i is $-i$; but you could also use i)

$$= \frac{(4 - 2i)(-i)}{(i)(-i)}$$

$$= \frac{-4i + 2i^2}{-i^2}$$

$$= \frac{-4i + 2(-1)}{-(-1)} \qquad (i^2 = -1)$$

$$= \frac{-4i - 2}{+1}$$

$$= -2 - 4i$$

Lesson 41 Exercises

A Find the square root or simplify:

1. $\sqrt{-9}$; **2** $\sqrt{-49}$; **3**. $\sqrt{-36}$; **4**. $\sqrt{-18}$; **5**. $\sqrt{-\frac{4}{9}}$; **6**. i^{40}; **7**. i^{93} ; **8**. i^{100}
9. $\sqrt{-28}$; **10**. $\sqrt{-32}$

Answers: **1**. $3i$; **2**. $7i$; **3**. $6i$; **4**. $3i\sqrt{2}$; **5**. $\frac{2}{3}i$; **6**. 1 ; **7**. i ; **8**. 1 ; **9**. $2i\sqrt{7}$; **10**. $4i\sqrt{2}$

B Simplify: **1**. $4 + 3i - 6 + 5i$; **2**. $(7 - 8i) + (2 + 9i)$ **3**. $(2 - 3i) - (7 - 4i)$;
4. Subtract $(1 - i)$ from $4 - 2i$; **5**. $4 + 2i - 3(4 - 6i) + i$; **6**. $-i^5 + i^2$; **7**. $4 - i + i^2$

Answers: **1**. $-2 + 8i$; **2**. $9 + i$; **3**. $-5 + i$; **4**. $3 - i$; **5**. $-8 + 21i$; **6**. $-1 - i$; **7**. $3 - i$

C Evaluate the following:
1. $(-2i)^2$; **2**. $6i^2$; **3**. $(-2i)(3i)$; **4**. $(-5i)(-6i)$; **5**. $(-i)(-2i)$; **6**. $i^3(-4i^3 + 2i^2)$

Answers: **1**. -4; **2**. -6; **3**. 6; **4**. -30; **5**. -2; **6**. $4 + 2i$..

D **Multiply**: $4 + 3i$ and $2 + 5i$; **2.** Multiply $2 - 5i$ and $-2 + 6i$; **3. Simplify**: $(6 - 7i)(2 + 3i)$;
4. Simplify $(1 - i)(1 + i)$;**5**. $(2 + 3i)^2$; **6**. $(4i^2)(-8i)(i^2)$; **7**. $(5 + 3i)(5 - 3i)$

Answers: **1.** $-7 + 26i$; **2.** $26 + 22i$; **3.** $33 + 4i$; **4.** 2 ; **5**. $-5 + 12i$; **6**. $-32i$; **7**. 34.

E Find the product of each of the following:
1. $(\sqrt{-4})(\sqrt{-1})$; **2**. $(\sqrt{-16})(\sqrt{-4})$; **3**. $(\sqrt{-25})(\sqrt{49})$; **4**. $(\sqrt{-8})(\sqrt{-2})$

Answers: **1**. -2; **2**. -8; **3**. $35i$; **4**. -4.

F Find the complex conjugate of each of the following:
1. $3 + 2i$; **2**. $4 - 5i$; **3**. $6i$; **4**. $-9i$; **5**. $-5 - 3i$; **6**. $-5 + 4i$.

Answers: **1**. $3 - 2i$; **2**. $4 + 5i$; **3**. $-6i$; **4**. $9i$; **5**. $-5 + 3i$; **6**. $-5 - 4i$..

G Divide: **1**. $\frac{8 - 6i}{2}$; **2**. $\frac{6 + 8i}{2i}$; **3**. $\frac{\sqrt{16}}{\sqrt{-16}}$; **4**. $\frac{3}{4 - \sqrt{-9}}$; **5**. $\frac{4 - 2i}{3 + 5i}$

Answers: **1**. $4 - 3i$; **2**. $4 - 3i$; **3**. $-i$; **4**. $\frac{12}{25} + \frac{9}{25}i$; **5**. $\frac{1}{17} - \frac{13}{17}i$

H Simplify: **1**. $\frac{2 + 3i}{5 - 2i}$; **2**. $\frac{4}{3 - 4i}$; **3**. $\frac{5 + 2i}{-i}$ **4.** Divide $4 + 3i$ by $-2 + 3i$;
Simplify: **5**. $\frac{2 - 6i}{4 + 3i}$; **6**. $\frac{i - 2}{2 - i}$; **7**. $(3 - 2i)(4 + 2i)(i^2)$; **8**. $(6 - 4i)(6 + 4i)$

Answers: **1**. $\frac{4}{29} + \frac{19}{29}i$; **2**. $\frac{12}{25} + \frac{16}{25}i$; **3**. $-2 + 5i$; **4**. $\frac{1}{13} - \frac{18}{13}i$; **5**. $-\frac{2}{5} - \frac{6}{5}i$; **6**. -1 ; **7**. $-16 + 2i$; **8**. 52

I Find the quotient and simplify: $\frac{(2 + 5i)(1 - i)}{(4 - 2i)(3 + i)}$

Answer: $\frac{23 + 14i}{50}$ or $\frac{23}{50} + \frac{7}{25}i$

Lesson 42

Equality of Complex Numbers; Roots of Complex Numbers; Equations Involving Complex Numbers

Equality of two complex numbers

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ (that is the real parts are equal) and $b = d$ (that is the imaginary parts are equal).

Example 4 Find x and y if $2x + 5yi = 8 + 15i$

Solution

Step 1: Set the real parts equal to each other and solve for x.

Then $2x = 8$
$x = 4$

Step 2: Set the imaginary parts equal to each other and solve for y.

Then $5y = 15$ and $y = 3$

Therefore $x = 4, y = 3$.

Roots of Complex Numbers

Example 5 Find the square root of 5 - 12i

Step 1 Let $x + yi$ be the square root of 5 - 12i (Where x and y are real numbers; Try a nd b instead)

Then $(x + yi)^2 = 5 - 12i$

$x^2 + 2xyi + y^2i^2 = 5 - 12i$

$x^2 - y^2 + 2xyi = 5 - 12i$ $\quad (i^2 = -1 \text{ and } y^2i^2 = y^2(-1) = -y^2)$

We apply the equality property of two complex numbers:

For the real parts: $x^2 - y^2 = 5$ (2)

For the imaginary parts:: $2xy = -12$ (3)

We shall now solve equations (2) and (3) simultaneously.
From equation (3),

$$y = -\frac{12}{2x}$$

$$y = -\frac{6}{x}$$

Substituting for y in equation (2)

$$x^2 - \left(-\frac{6}{x}\right)^2 = 5$$

$$x^2 - \frac{36}{x^2} = 5$$

$$x^4 - 36 = 5x^2$$

$$x^4 - 5x^2 - 36 = 0$$

We shall use the quadratic formula to solve for x. .

Step 2:: By the substitution method (Also, see p.38)

Let $x^2 = u$

Then $x^4 - 5x^2 - 36 = 0$ becomes

$u^2 - 5u - 36 = 0 \qquad (u = x^2)$

Solving, $u = \dfrac{5 \pm \sqrt{169}}{2}$

$u = \dfrac{5 \pm 13}{2}$

Now, converting back to x by letting $u = x^2$

$x^2 = \dfrac{5 \pm 13}{2}$

$x^2 = \dfrac{5 + 13}{2}$ or $x^2 = \dfrac{5 - 13}{2}$

$x^2 = \dfrac{18}{2}$ or $x^2 = \dfrac{-8}{2}$

$x^2 = 9$ or $x^2 = -4$

$x = \pm\sqrt{9}$ or $x = \sqrt{-4}$

$x = \pm 3$ or $x = \pm 2i$

We reject the imaginary solution since x and y must be real.

Therefore, $x = \pm 3$.

Step 3: Substitute $x = \pm 3$ in $y = -\dfrac{6}{x}$ to obtain the corresponding y-values.

When $x = +3$, $y = -\dfrac{6}{3} = -2$, and

when $x = -3$, $y = -\dfrac{6}{-3} = 2$.

Thus, when $x = 3$,, $y = -2$, and
when $x = -3$, $y = 2$.

Substituting these values in $x + yi$, the two square roots are $(3 - 2i)$ and $(-3 + 2i)$.

These roots can be verified by squaring each root.

Note above that **in Step 2,** we could have solved the quadratic equation by factoring as follows:

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

$x^2 - 9 = 0$; or $x^2 + 4 = 0$

$x^2 = 9$; or $x^2 = -4$

$x = \pm 3$ or $x = \pm 2i$;

This approach would be faster than by substitution.

Example 6 If $z^2 = 8i$, find z satisfying this equation.

Solution: $z = \pm\sqrt{8i}$

Thus, we are to find the square root of $8i$. We can use the same method as in Example 5 , above.

Let $a + bi$ be the square root of $8i$. (where a and b are real)

Then $(a+bi)^2 = 8i$ (By definition, $\sqrt{8i} = a + bi$ if $(a+bi)^2 = 8i$)

$a^2 + 2abi - b^2 = 8i + 0$ (Note: $b^2i^2 = b^2(-1) = -b^2$)

$a^2 - b^2 + 2abi = 8i + 0$ (Also, note that $(a^2 - b^2,\ 2abi) = (0,\ 8i)$)

Equating the real parts, $a^2 - b^2 = 0$ (1)

Equating the imaginary parts, $2ab = 8$ (2)

We shall now solve equations (1) and (2) simultaneously for a and b.

From equation (2), $b = \dfrac{4}{a}$ (3)

Substituting for b from (3) in equation (1),

$$a^2 - \left(\frac{4}{a}\right)^2 = 0$$

$$a^2 - \frac{16}{a^2} = 0$$

$$a^2 \bullet a^2 - \frac{16 \bullet a^2}{a^2} = 0 \bullet a^2$$ (multiplying each term of the equation by a^2 to undo the denominator)

$$a^4 - 16 = 0$$

$$a^4 = 16$$

$$a^2 = \pm 4$$

$$a^2 = +4 \quad \text{or } a^2 = -4$$

$$a = \pm 2 \quad \text{or} \quad a = \pm 2i.$$

Since a must be a real number, we reject the imaginary a-values, $\pm 2i$, and accept the real a-values, ± 2.
Therefore a = +2 or -2

When a= +2, $b = \dfrac{4}{2} = 2$

When a= -2, $b = \dfrac{4}{-2} = -2$

Substituting the values of a and b in $a + bi$, the two square roots of $8i$ are $(2 + 2i)$ and $(-2 - 2i)$.
Verify these roots by squaring each root.

Note: In the above problem, do not confuse $\pm\sqrt{8i}$ with $\pm\sqrt{-8}$ which is $\pm i\sqrt{8}$ or $\pm 2i\sqrt{2}$.

Equations involving complex numbers

Example 7 Solve for x, given that
$$3ix + 12 - 8i + 2x = 2(6 - i) + 2x$$

Solution We shall get x alone on one side of the equation and the other side should not contain x:

$$3ix + 12 - 8i + 2x = 2(6 - i) + 2x$$
$$3ix - 8i + 12 + 2x = 12 - 2i + 2x$$
$$3ix - 8i = -2i$$
$$(3x - 8)i = -2i$$
$$3x - 8 = -2$$
$$x = 2$$

Example 8 Solve for x, given that $2ix + 3 - 4i = (3 + 2i)x + 5i$

Solution We shall get x alone on one side of the equation and the other side should not contain x:

$$2ix + 3 - 4i = (3 + 2i)x + 5i$$
$$2ix + 3 - 4i = 3x + 2ix + 5i$$
$$2ix - 3x - 2ix = -3 + 9i$$
$$2ix - 3x - 2ix = -3 + 9i$$
$$-3x = -3 + 9i$$
$$x = 1 - 3i$$

Example 9 Solve for x, if x is a real number, given that

Solution

$$2ix + 3 - 4i = (3 + 2i)x + 5i$$
$$2ix + 3 - 4i = 3x + 2ix + 5i$$
$$2ix - 3x - 2ix = -3 + 9i$$
$$2ix - 3x - 2ix = -3 + 9i$$
$$-3x = -3 + 9i$$
$$x = 1 - 3i$$

Since on solving for x, the solution contains the imaginary unit, x is not real.
Therefore, there is **no** solution. (From the original problem, for a solution, x must be real)
.

Lesson 42 Exercises

A Find the square roots of the following **1**. $16 - 30i$; **2**. $-5 - 12i$

Answers: **1**. $5 - 3i$ and $-5 + 3i$; **2**. $2 - 3i$ and $-2 + 3i$

B Find z satisfying the given equation **1**. $z^2 = 16 - 30i$; **2**. $z^2 = -5 - 12i$

Answers: **1**. $5 - 3i$ and $-5 + 3i$; **2**. $2 - 3i$ and $-2 + 3i$

C Solve for x: **1**. $3ix - 30 + 3i = (12 + 3i)x + 3i$; **2**. $x^2 = 16i$

3. Determine the real values of x and y if $4x + 7yi = 12 - 35i$

Answers: **1.** $x = -\frac{5}{2}$; **2**. $x = 2\sqrt{2} + 2i\sqrt{2}$, $x = -2\sqrt{2} - 2i\sqrt{2}$; **3**. $x = 3$, $y = -5$.

Lesson 43
Graphical Representation and Addition of Complex Numbers

A complex number may be considered as an ordered pair of real numbers (a, b). Graphically, we represent a complex number z by a point P whose x-coordinate = a, and y-coordinate = b. We graph a complex number in a rectangular coordinate system of axes. We call this system the complex plane. The horizontal axis (x-axis) represents the real axis and the vertical axis (y-axis) represents the imaginary axis.

Example: Graph the following in the complex plane:

(a) $2-3i$; (b) $3+2i$; (c) $-4-3i$; (d) $-i$; (e) $5i$; (f) 6.

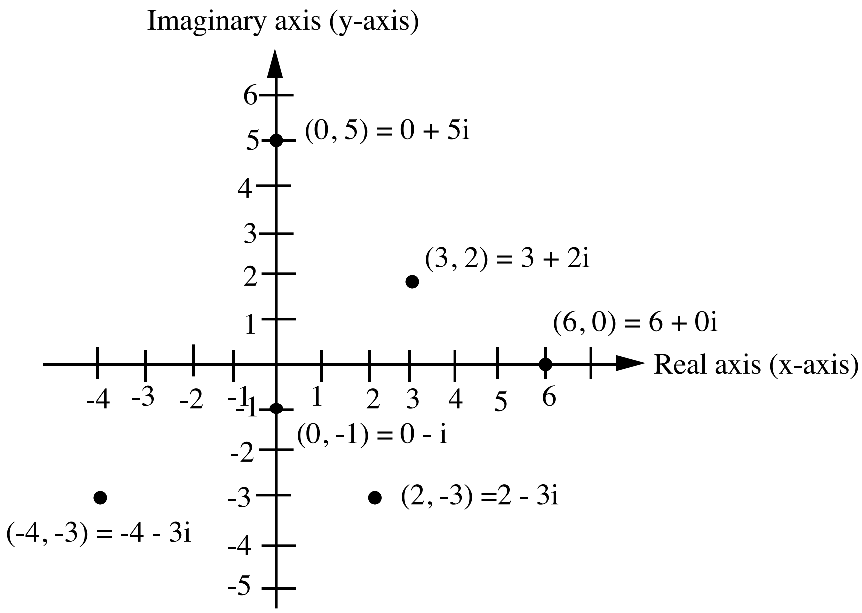

Figure: The graphs of the points, $2-3i$, $3+2i$, $-4-3i$, $-i$, $5i$, 6, in the complex plane.

We may also think of a complex number as a vector. We represent a vector quantity by an arrow drawn to scale. The length of the arrow represents the magnitude of the vector and the direction of the arrow head represents the direction of the vector. In Figure we represent the complex number $a+bi$ as a vector drawn from the origin to the point $(a, b) = a + bi$..

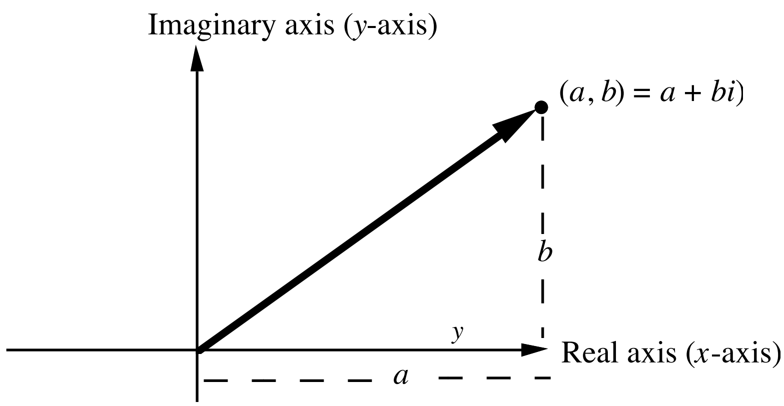

Figure: Graphical representation of a complex number as a vector.

Graphical Addition of Complex Numbers

We may use the parallelogram law of **vector addition** to add complex numbers.

In the complex plane, we draw each of the two vectors $a + bi$ and $c + di$ to scale. The **sum** of the complex numbers is represented by the **diagonal** of the parallelogram (Figure).

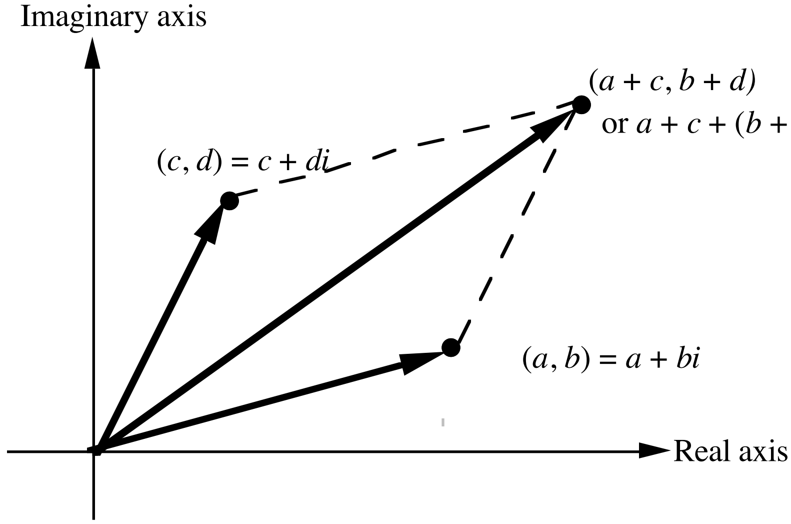

Figure: Graphical addition of complex numbers: $(a + bi) + (c + di)$

Subtraction of Complex Numbers: $(a+bi)-(c+di)=(a+bi)+(-c-di)$

By definition, to subtract $(c+di)$ from $(a+bi)$ means add the negative of $(c + di)$ to $(a + bi)$. Graphically, $(-c-di)$ has the same magnitude as $(c+di)$ except that its direction is opposite to that of $(c+di)$ as shown in Figure

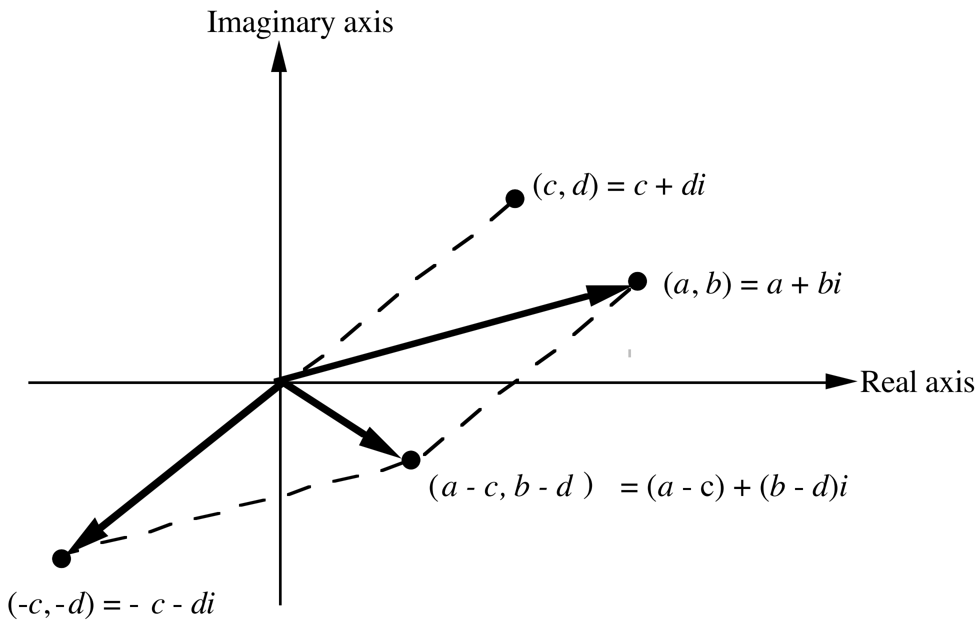

Figure: Graphical subtraction of complex numbers: $(a + bi) - (c + di)$

Example 1 Add algebraically and graphically: $(2+3i)+(7+5i)$.

Algebraic method

$$(2+3i)+(7+5i)$$
$$= 2+3i+7+5i$$
$$= 9+8i$$

Graphical method

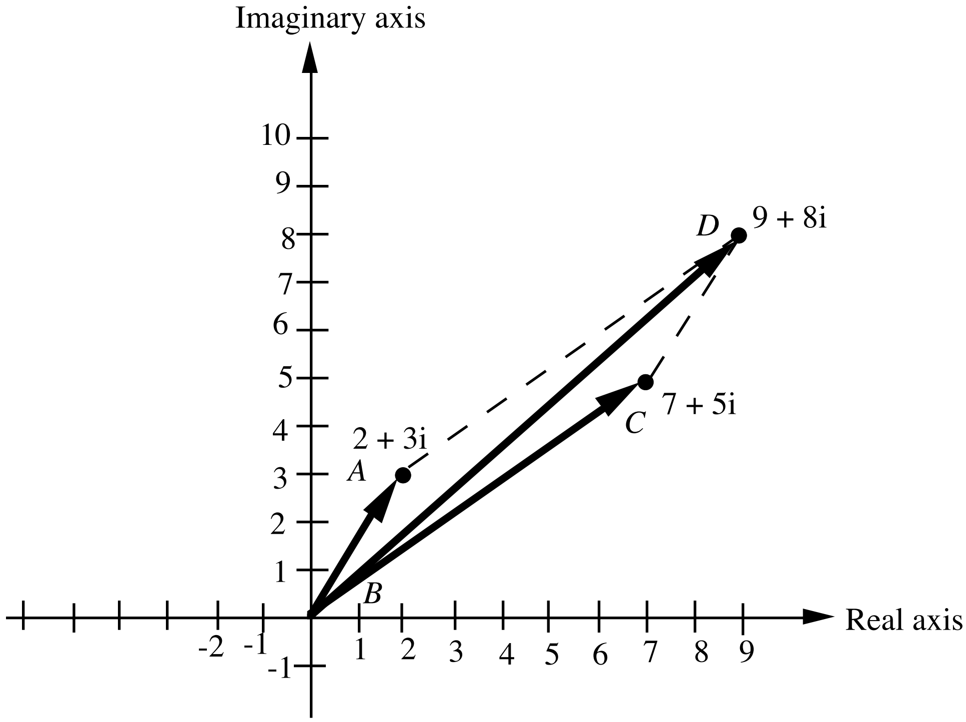

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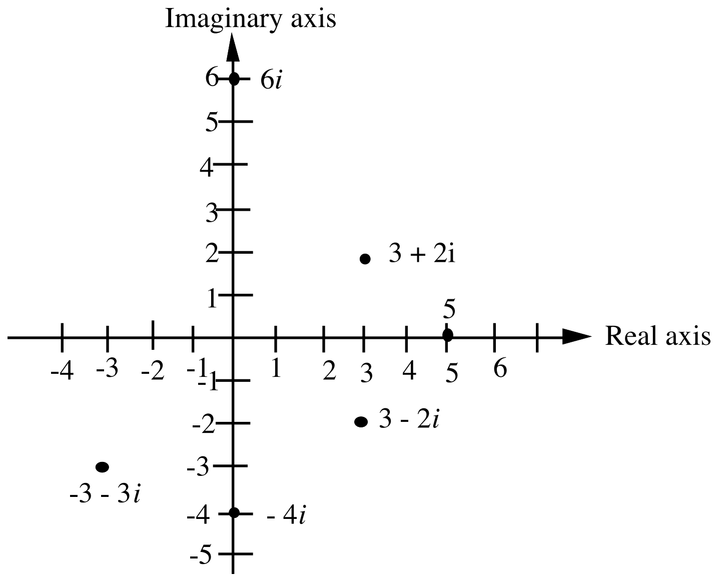
Figure : $(2+3i)+(7+5i)$ is represented by *BD*

Lesson 43 Exercises

Graph each of the following in the complex plane:

1. $3-2i$; **2**. $3+2i$; **3**. $-3-3i$; **4**. $-4i$; **5**. $6i$; **6**. 5

Solution



Lesson 44

Polar (Trigonometric) Representation of Complex Numbers.

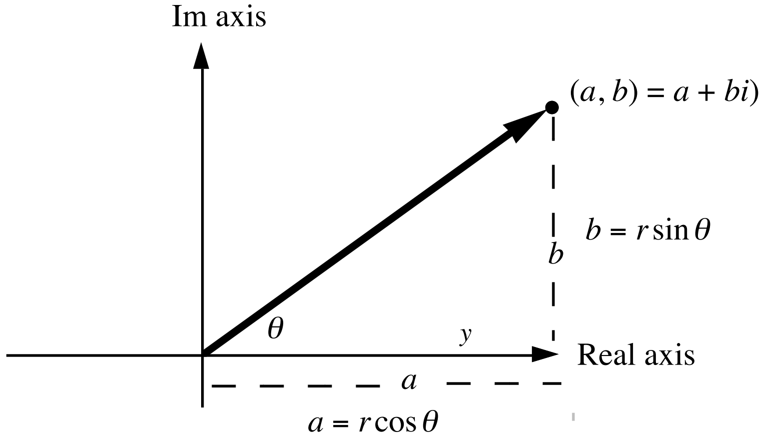


Figure : The graph of polar or trigonometric form of a complex number.

A complex number $z = a + bi$ is represented by a vector drawn from the origin to the point (a, b) and in direction θ with the x-axis. The standard position angle, θ, is called the argument of the complex number.

The **length** r of the vector is given by $r = \sqrt{a^2 + b^2}$

From trigonometric definitions,

$$a = r\cos\theta \qquad (1)$$

$$b = r\sin\theta \qquad (2)$$

$$\theta = \arctan\left(\frac{b}{a}\right) \qquad (3)$$

By substituting for a and b from equations (1) and (2) respectively, in the rectangular form, $z = a + bi$,we obtain

$z = r\cos\theta + r\sin\theta i$ or

$z = r\cos\theta + ir\sin\theta$; and if we factor out the r, we obtain

$$\boxed{z = r(\cos\theta + i\sin\theta)} \qquad (4)$$

Sometimes, the quantity $(\cos\theta + i\sin\theta)$ is abbreviated by cis θ and then equation (4) becomes

$$z = rcis\theta \qquad (5)$$

We can view this abbreviation this way:

$$z = r\,(\,\cos\theta + i\ \sin\theta\,)$$
$$\uparrow\ \uparrow\ \uparrow\ \ \uparrow\uparrow\ \uparrow$$
$$z = r\ c\ \ i\ s$$

Although it might be cumbersome to write equation (4) repeatedly instead of equation (5), it is recommended that we use the form of equation (4) a number of times before switching to the abbreviated form, $z = rcis\theta$.

Changing from rectangular form to polar form

Rectangular form: $z = a + bi$
Polar form: $z = r(\cos\theta + i\sin\theta)$ or $z = rcis\theta$

$$r = \sqrt{a^2 + b^2}$$

$$a = r\cos\theta \qquad (1)$$

$$b = r\sin\theta \qquad (2)$$

$$\theta_{ref} = \arctan\left(\frac{b}{a}\right) \qquad (3)$$

Example 1 Change the complex number $z = 3 - 4i$ to polar form.

Solution In polar form, we want the form $z = r(\cos\theta + i\sin\theta)$. So we must find r and θ from the rectangular form.

Step 1: Plot $z = 3 - 4i$ as a vector (Figure)

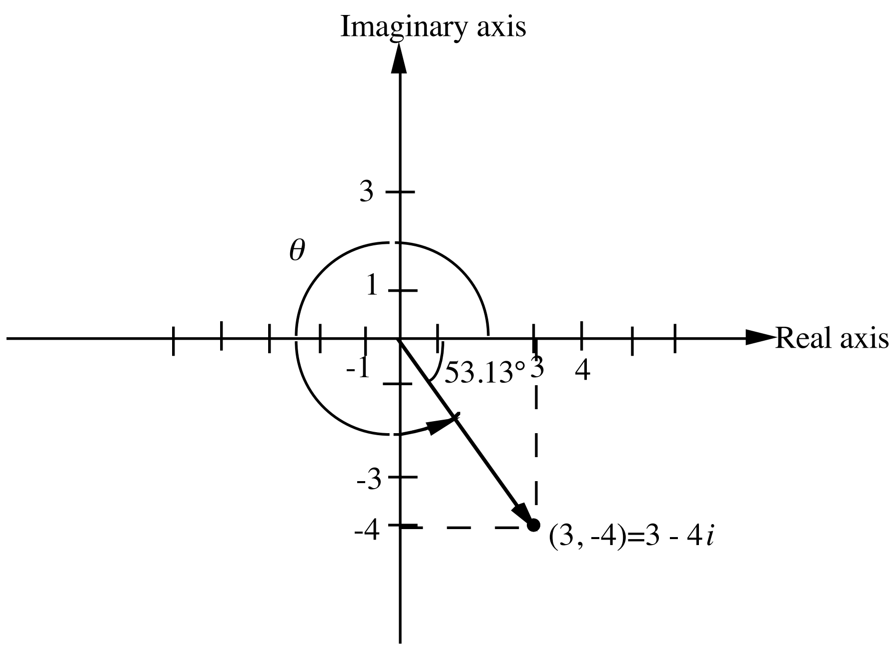


Figure: The graph of $z = 3 - 4i$

Step 2: Find r.

$a = 3, b = -4$

$$r = \sqrt{3^2 + (-4)^2}$$

$$r = \sqrt{9 + 16}$$

$$r = \sqrt{25}$$

$$r = 5.$$

Step 3: Find θ.

$\tan\theta_{ref} = \frac{b}{a}$ (θ_{ref} is the reference angle)

$= -\frac{4}{3}$

$= -1.3$

Now, from tables or calculator, find the angle whose tangent is 1.3

$\theta_{ref} = 53.13°$ (Terminal side of θ is in the 4th quadrant)

$\theta = 360° - 53.13°$

$\theta = 306.87°$

(**Note** above that there is a difference between θ and θ_{ref})

Step 4: Substitute $r = 5, \theta = 305.87°$ in $z = r(\cos\theta + i\sin\theta)$.

Then $z = 5(\cos 306.87° + i\sin 306.87°)$

or $z = 5cis306.87°$.

We must note above that the polar form was **not** specified in terms of $53.13°$ but rather in terms of $306.87°$, this angle being in standard position, and as such, it is measured counter-clockwise from the positive *x*-axis. ($53.13°$ is the reference angle from tables or a calculator.)

Example 2 Change to polar (trigonometric) form: $z = -3 - 5i$

Step 1: Plot $z = -3 - 5i$ as a vector (Figure)

Step 2: Find *r*.

$a = -3, b = -5$ (from $z = -3 - 5i$) (**Note**: General form is $z = a + bi$)

$r = \sqrt{(-3)^2 + (-5)^2}$ $(r = \sqrt{a^2 + b^2})$

$r = \sqrt{9 + 25}$

$r = \sqrt{34}$

Step 3: Find θ

$\tan\theta_{ref} = \frac{-5}{-3}$ $\left(\frac{b}{a}\right)$

$= 1.67$

$\theta_{ref} = 59.03°$ (Now, from tables or calculator, find the angle whose tangent is 1.67)

$\theta = \theta_{ref} + 180°$ (The terminal side of θ is in the 3rd quadrant.)

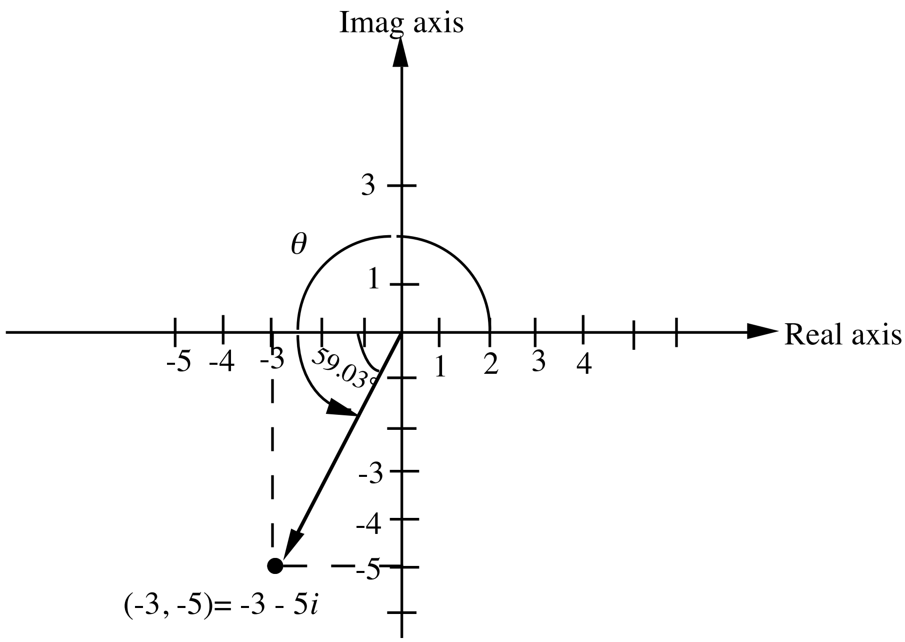
$\theta = 59.03° + 180°$

$\theta = 239.03°$

Step 4: Substitute $r = \sqrt{34}$, $\theta = 239.03°$ in

$z = r(\cos\theta + i\sin\theta)$ to obtain

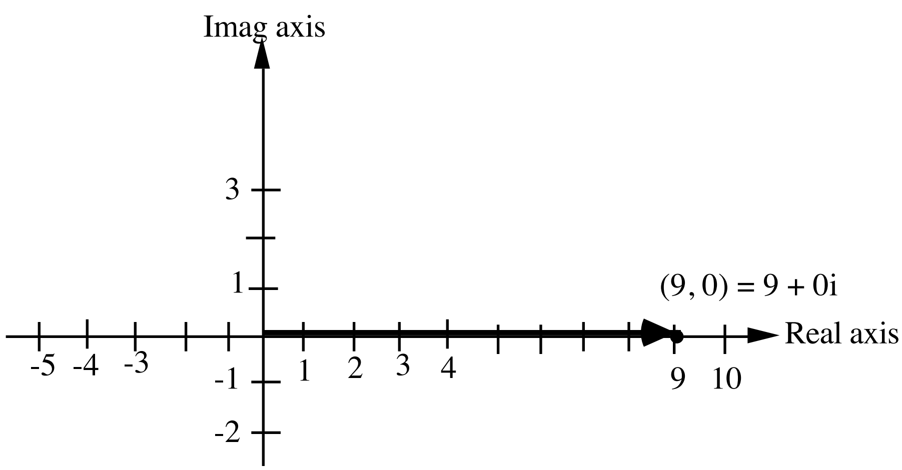
$z = \sqrt{34}(\cos 239.03° + i\sin 239.03°)$. <------polar (or trigonometric) form



Example 3 Change to polar form.
(a) 9 ; (b) -9 ; (c) $-3i$.

Solution

(a) Step 1: Sketch $z = 9 + 0i$ (Figure)



Step 2: Find r

$r = 9$ (since θ is quadrantal; terminal side is along the positive x-axis)
$a = 9$
$b = 0$

Step 3: Find θ

$$\tan\ \theta_{ref} = \frac{0}{9} = 0$$

$\theta_{ref} = 0°$

In this problem, $\theta = \theta_{ref}$.

$\theta = 0°$.

Step 4: Substitute $r = 9$, $\theta = 0°$ in the general equation ($z = r(\cos\theta + i\sin\theta)$) to obtain

$z = 9(\cos 0° + i\sin 0°)$

$z = 9\cos 0°$ or $9\,cis 0°$ $\qquad$ ($\sin 0° = 0$)

(b) Step 1: Sketch $z = -9 + 0i$ (Figure)

Step 2: Find r and θ.

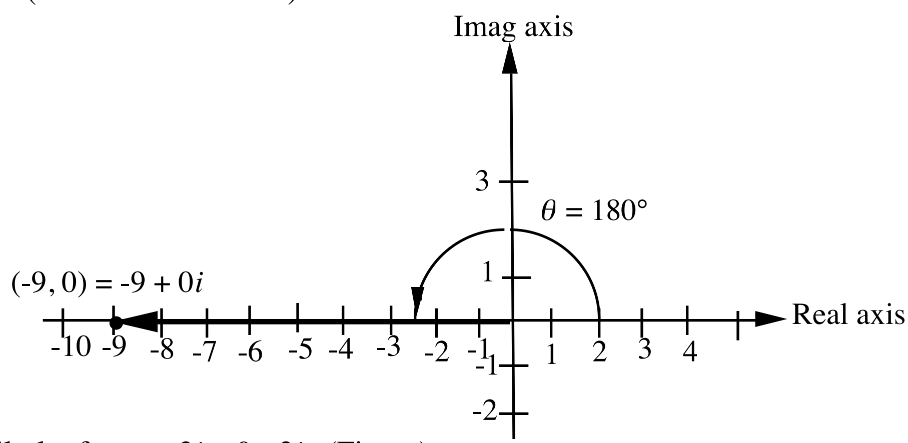
$r = 9$ $\qquad$ (Note that r is positive.)

By inspection, since the terminal side falls along the negative x-axis, $\theta_{ref} = 0°$

Therefore, $\theta = 180°$

Substituting $r = 9$, $\theta = 180°$ in the general equation, we obtain

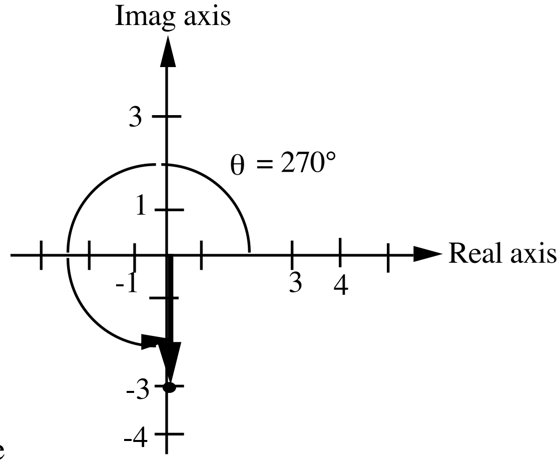
$z = 9(\cos 180° + i\sin 180°)$



(c) Similarly, for $z = -3i = 0 - 3i$ (Figure)

$r = 3$, $\theta = 270°$ and

$z = 3(\cos 270° + i\sin 270°)$ or $z = 3\,cis 270°$



Figure

We must note above that by agreement , we restricted the angular measurement to $0° \le \theta \le 360°$.

Changing from polar form to rectangular (algebraic) form

Example 1 Change $z = 10(\cos 220° + i \sin 220°)$ to rectangular form.

The general rectangular form is given by

$$z = a + bi.$$

We must therefore find a and b from the polar form.

By inspection and comparison of $z = r(\cos\theta + i\sin\theta)$,with

$$z = 10(\cos 220° + i \sin 220°)$$

$$r = 10, \quad \theta = 220°$$

$$a = 10\cos 220° \qquad (a = r\cos\theta)$$

$$a = -7.7 \qquad (\cos 220° = -.77)$$

$$b = 10\sin 220°$$

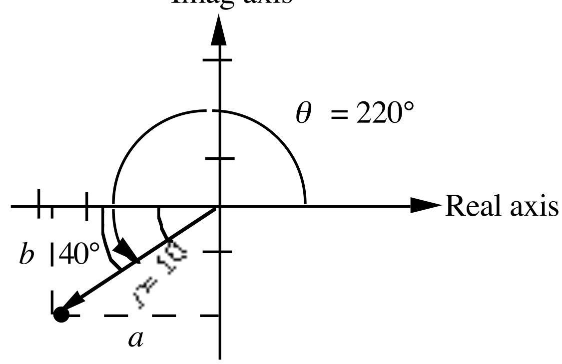
$$b = -6.4 \quad (\sin 220° = -.64)$$

Substituting $a = -7.7,\ b = -6.4$ in $z = a + bi$, we obtain

$$z = -7.7 - 6.4i$$

$$10\, cis\, 220° = -7.7 - 6.4i$$

↑ (Polar form) ↑ (Rectangular form)



Figure

It may be remarked that it easier to change from the polar form to the rectangular form than vice versa

Exponential form of a complex Number

The exponential form of a complex number, z, is given by $z = re^{i\theta}$, where θ is in radians.

Multiplication of Complex Numbers in Polar Form

Consider the polar (trigonometric) forms of two complex numbers,

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos_2 + i\sin\theta_2)$.

The product $z_1 z_2$ of the two complex numbers in polar form is given by

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad (1)$$

Therefore, to multiply two complex numbers in polar forms, multiply the radii (moduli) and add the angles (arguments).

Example Given that $z_1 = 3(\cos 30° + i\sin 30°$
$z_2 = 4(\cos 40° + i\sin 40°)$

(a) Find by analytic means (using equation (1) above) the product of z_1 and z_2.

(b) Show the graphical form of the above method.

Solution

(a) By equation (1) above,

Apply $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta)]$ with
$r_1 = 3,\ r_2 = 4,\ \theta_1 = 30°,\ \theta_2 = 40°$ to obtain

$$z_1z_2 = (3)(4)[\cos(30° + 40°) + i\sin(30° + 40°)$$
$$= 12[\cos 70° + i\sin 70°] \qquad (2)$$
$$z_1z_2 = 12cis70°.$$

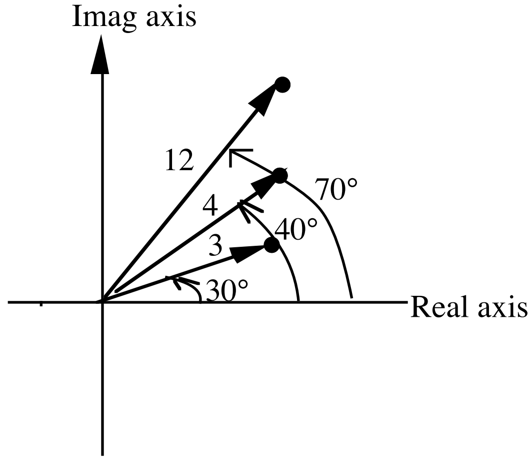
(From tables or calculator we can simplify equation (2) for z_1z_2)

(b) Graphical form:

Step 1: Find the sum of the angles $30°$ and $40°$
Then $\theta = 70°$

Step 2: Find the product $r = r_1r_2$
Then $r = (3)(4) = 12$

Step 3: Draw $\theta = 70°$ in standard position and $r = 12$ (Figure)



Division of Complex Numbers in Polar Form

Consider the polar (trigonometric) form of two complex numbers, say

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos_2 + i\sin\theta_2)$.

The quotient $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

or $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\ \text{cis}(\theta_1 - \theta_2)$.

Therefore, to divide two complex numbers in polar forms, divide the radii (moduli) and subtract the angles (arguments).

Example l Given that $z_1 = 15(\cos 50° + i\sin 50°)$ and

$z_2 = 5(\cos 30° + i\sin 30°)$, find the quotient $\frac{z_1}{z_2}$

Solution

$$\frac{z_1}{z_2} = \frac{15}{5}[\cos(50° - 30°) + i\sin(50° - 30°)]$$
$$= 3[\cos 20° + i\sin 20°] \text{ or } 3\text{cis}\,20°$$

Example 2 If $z_1 = 6(\cos 15° + i\sin 15°)$ and $z_2 = 12(\cos 50° + i\sin 50°)$ find the quotient $\frac{z_1}{z_2}$.

Solution

$$\frac{z_1}{z_2} = \frac{6}{12}[\cos(15° - 50°) + i\sin(15° - 50°)]$$
$$= \frac{1}{2}[\cos(-35) + i\sin(-35)] \text{ or } \frac{1}{2}\text{cis}(-35°)$$

But since we want to specify θ so that $0° \le \theta \le 360°$

$$\frac{z_1}{z_2} = \frac{1}{2}\text{cis}(325°)$$

Lesson 44 Exercises

A Plot each complex number and then change to polar form.

1. $3+4i$; **2**. $1+i$; **3**. $1-3i$; **4**. $3-i$; **5**. $1-i\sqrt{3}$; **6**. 7;
7. $-i$; **8**. $-3-4i$; **9**. $-5i$; **10**. -3; **11**. $-.69+3.94i$

Answers:

1. $z = 5cis53.13°$; 2. $z = \sqrt{2}\,cis45°$; **3**. $z = \sqrt{10}\,cis288.43°$; **4**. $z = \sqrt{10}\,cis341.57°$; **5**. $z = 2cis300°$; **6**. $z = 7cis0°$; **7**. $z = cis270°$; **8**. $z = 5cis233.13°$; **9**. $z = 5cis270°$; **10**. $z = 3cis180°$; **11**. $4cis100$

B Plot each complex number and then change to rectangular form.

1. $4(\cos 30° + i\sin 30°)$; **2**. $2(\cos 210° + i\sin 210°)$; **3**. $5cis30°$;
4. $3(\cos 0° + i\sin 0°)$; **5**. $2cis150°$; **6**. $9cis\frac{2\pi}{3}$.

Answers:

1. $z = 2\sqrt{3} + 2i$; 2. $z = -\sqrt{3} - i$; **3**. $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$; **4**. $z = 3$; **5**. $z = -\sqrt{3} + i$; **6**. $z = -\frac{9}{2} + \frac{9i\sqrt{3}}{2}$

C Perform the indicated operation and leave the answer in polar form.

1. $(\cos 20° + i\sin 20°)(\cos 10° + i\sin 10°$; **2**. $3(\cos 25° + i\sin 25) \bullet 4(\cos 125° + i\sin 125°)$;
3. $[2(\cos 60° + i\sin 60°)]^3$; **4**. $(3cis45°)(4cis360°)$

Answers: **1**. $z = cis30°$; 2. $z = 12cis150°$; **3**. $z = 8cis180°$; **4**. $z = 12cis405°$.

D Perform the indicated operations and leave the answers in polar form.

1. $\frac{14cis170°}{2cis60°}$; **2**. $\frac{10cis60°}{5cis\,90°}$ **3**. $\frac{15(\cos 100° + i\sin 100°)}{3(\cos 25° + i\sin 25°)}$

Answers: **1**. $z = 7cis110°$; 2. $z = 2cis330°$; **3**. $z = 5cis75°$.

Lesson 45

Powers of Complex Numbers; De Moivre's Theorem; and Roots of Complex Numbers

Powers of Complex Numbers

Let us consider the squaring $z = r\text{cis}\,\theta$

By the multiplication rule, we multiply the moduli (radii) and add the arguments (angles).

Thus $z^2 = (r\text{cis}\,\theta)^2$

$= (r\text{cis}\,\theta)(r\text{cis}\,\theta)$

$= (r)(r)\text{cis}(\theta + \theta)$

$= r^2\text{cis}\,2\theta$

Similarly,

$z^3 = r^3\text{cis}\,3\theta$

$z^4 = r^4\text{cis}\,4\theta$

From the above examples, we arrive at a generalization known as **De Moivre's** theorem or formula.

De Moivre's Theorem: If n is a positive integer, then

$$z^n = (r\,\text{cis}\theta)^n = r^n\text{cis}\,n\theta$$

or $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

Example 1 Use De Moivre's theorem to find $(3-4i)^6$.

Step 1: Change the complex number to polar (trigonometric) form.
Thus, we want to change 3 - 4*i* to the form $z = r(\cos\theta + i\sin\theta)$

Step 2: Previously, $(3-4i) = 5(\cos 306.87° + i\sin 306.87°)$

Applying $z^n = r^n(\cos n\theta + i\sin n\theta)$, we obtain

$(3-4i)^6 = [5(\cos 306.87° + i\sin 306.87°)^6$

$= 5^6[\cos(6)(306.87°) + i\sin(6)(306.87°)]$

$= 5^6[\cos 1841.22° + i\sin 1841.22°]$

$= 15625[\cos 41.22° + i\sin 41.22°]$

$= 15625\,\text{cis}\,41.22°$ <--- polar form

In exponential form:: $z = r\,e^{i\theta}$, where θ is in radians

$$41.22^\circ = \frac{41.22^\circ}{180^\circ} \times \frac{\pi \text{ rad}}{1} = 0.72 \text{ rad}$$

$(3-4i)^6. = 15625e^{0.72i}$ <----exponential form ($r = 15625$,$\theta = 0.72$ rad)

$(3-4i)^6. = 11753 + 10296i$ **<---** rectangular form

Comparison

Rectangular form	**Polar form**	**Exponential form**
$z = a + bi$ or $z = x + iy$	$z = r(\cos\theta + i\sin\theta = r\text{cis}\theta$	$z = r\,e^{i\theta}$

Roots of Complex Numbers

We may apply De Moivre's theorem to find the roots of complex numbers.

if $x^n = A$, (where n is a positive integer)

then $x = A^{\frac{1}{n}}$

Example: If $x^3 = 8$, then $x = \sqrt[3]{8} = 8^{\frac{1}{3}}$
Similarly, for a complex number A

if $x^n = A$, then $x = A^{\frac{1}{n}}$ (where n is a positive integer).
There are n distinct roots, where x is an nth root of A.

According to De Moivre's Theorem. (which deals with powers of complex numbers in polar form),

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \ .$$

If we replace n by $\frac{1}{n}$ in De Moivre's Theorem, and take into account the periodic nature of the cosine and sine functions, we obtain

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}[\cos\frac{(\theta + 360^\circ k)}{n} + i\sin\frac{(\theta + 360^\circ k)}{n}] \qquad \text{where } k = 0, 1, 2, \ldots$$

$$= \sqrt[n]{r}[\cos\frac{(\theta + 360^\circ k)}{n} + i\sin\frac{(\theta + 360^\circ k)}{n}]$$

$$(r\,\text{cis}\,\theta)^{\frac{1}{n}} = r^{\frac{1}{n}}\,\text{cis}\frac{(\theta + 360^\circ k)}{n}$$

$$= \sqrt[n]{r}\,\text{cis}\frac{(\theta + 360^\circ k)}{n}$$

If $n = 2$, there are two distinct roots and we shall use two k-values, namely, $k = 0, k = 1$.
For example, if $k = 0$,

$$\frac{\theta + 360^\circ k}{n} = \frac{\theta + 360^\circ(0)}{n}$$

$$= \frac{\theta + 0}{n}$$

$$= \frac{\theta}{n}.$$

If $k = 1$,

$$\frac{\theta + 360^\circ k}{n} = \frac{\theta + 360^\circ(1)}{n}$$

$$= \frac{\theta + 360^\circ}{n}.$$

Similarly, if $n = 3$, there shall be three distinct roots and we use three k-values, namely , $k = 0$, 1, and 2.

We should note that the k-values, $k = 0, 1, 2, \ldots, n - 1$ are used only in finding the roots of the complex numbers but are **not** used in computing the powers (De Moivre's Theorem).
For $k = n - 1$, the results repeat one of the first n values (previous values).

Applying De Moivre's Theorem to Find the Roots of Numbers

Step 1: Change the given expression to polar (trigonometric) form (if it is already not in polar form).

Step 2: Apply the formula

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = \sqrt[n]{r}[\cos\frac{(\theta + 360°k)}{n} + i\sin\frac{(\theta + 360°k)}{n}] \qquad \text{where } k = 0,1,2,...$$

with the appropriate k-values.

Example 1 Find the 4-th roots of $16(\cos 120° + i\sin 120°)$.

Solution

The wording of this problem is equivalent to:

if $z^4 = 16(\cos 120° + i\sin 120°)$, solve for z.

Since the given expression is already in polar form, we go ahead to Step 2 and apply the formula

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = \sqrt[n]{r}[\cos\frac{(\theta + 360°k)}{n} + i\sin\frac{(\theta + 360°k)}{n}] \qquad \text{with } k = 0,\ 1,\ 2, \text{ and } 3.$$

$n = 4,\ r = 16,\ \theta = 120°$

Substituting these values

$$z = \sqrt[4]{16}[\cos\frac{(120° + 360°k)}{4} + i\sin\frac{(120° + 360°k)}{4}]$$

$$z = 2[\cos\frac{(120° + 360°k)}{4} + i\sin\frac{(120° + 360°k)}{4}] \qquad \text{(A)}$$

We can leave the answer in either the trigonometric (polar) form or the rectangular form , $z = a + bi$

Since $n = 4$, we shall use k-values of $k = 0, 1, 2, 3$.

When $k = 0$ in (A), we obtain the root,

$$z = 2[\cos\frac{(120° + 0)}{4} + i\sin\frac{(120° + 0)}{4}]$$

$= 2[\cos 30° + i\sin 30°]$ <-------------------Polar form

$= 2[\frac{\sqrt{3}}{2} + \frac{1}{2}i]$

$= \sqrt{3} + i.$ <------------------------------Rectangular form

When $k = 1$ in (A), we obtain the root

$$z = 2[\cos\frac{(120° + 360°(1))}{4} + i\sin\frac{(120° + 360°(1))}{4}]$$

$= 2[\cos\frac{(120° + 360°}{4} + i\sin\frac{(120° + 360°)}{4}]$

$= 2[\cos\frac{480°}{4} + i\sin\frac{480}{4}]$

$= 2[\cos 120° + i\sin 120°]$ <-----------------polar form

$= 2[-\cos 60° + i\sin 60°]$

$= 2[-\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)]$

$= -1 + \sqrt{3}i$ or $-1 + i\sqrt{3}$ <------------------rectangular form

For $k = 2$ in (A), we obtain the root.

$z = 2[\cos\frac{(120° + 360°(2))}{4} + i\sin\frac{(120° + 360°(2))}{4}]$

$= 2[\cos\frac{(120° + 720°}{4} + i\sin\frac{(120° + 720°)}{4}]$

$= 2[\cos\frac{840°}{4} + i\sin\frac{840°}{4}]$

$= 2[\cos 210° + i\sin 210°]$ <----------------polar form

$= 2[-\cos 30° + i\sin 30°]$

$= 2[-\left(\frac{\sqrt{3}}{2}\right) + i\left(-\frac{1}{2}\right)]$

$= -\sqrt{3} - i$ <------------------rectangular form

Similarly, for $k = 3$ in (A) we obtain the root

$2[\cos 300° + i\sin 300°]$ <----------------polar form

$= 2[\cos 60° + i(-\sin 60°)]$

$1 - \sqrt{3}i$ or $1 - i\sqrt{3}$ <------------------rectangular form

The 4th roots of $16(\cos 120° + i\sin 120°)$ are $\sqrt{3} + i$; $-1 + i\sqrt{3}$; $-\sqrt{3} - i$; and $1 - i\sqrt{3}$

Example 2 If $z^4 = -8 + 8\sqrt{3}i$, solve for z.

Solution

Step 1: Change the right-hand side of the equation to polar form.

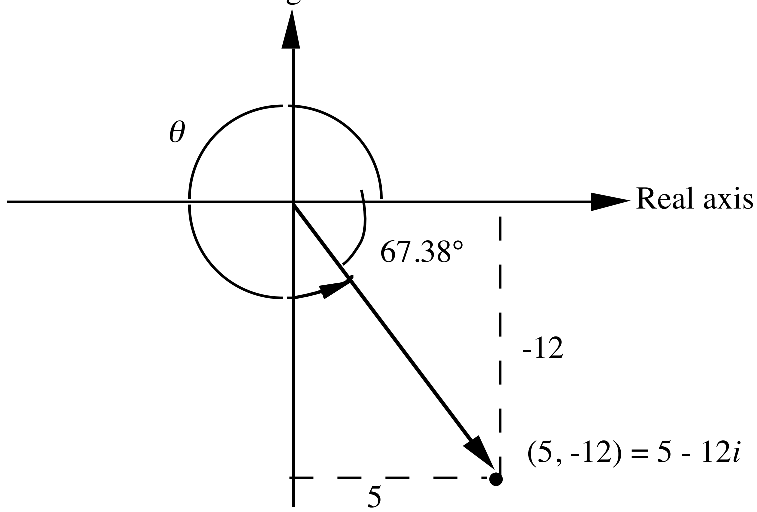
$r = 16, \quad \theta = 120°$

$-8 + 8\sqrt{3}i = 16(\cos 120 + i\sin 120)$

$\therefore\ z^4 = 16(\cos 120 + i\sin 120)$.

The rest of the steps are the same as those in Example 1 , above.

Example 3 (Method 1) Find the square root of $5-12i$ (This problem was done previously.



Step 1: Change to polar form.

$a = 5, b = -12$

$r = \sqrt{5^2 + (-12)^2} = 13$

$\tan\theta_{\text{ref}} = -\frac{12}{5}$

$\theta_{\text{ref}} = 67.38°$

$\theta = 360° - 67.38°$

$\theta = 292.62°$

Now, $r = 13$, $\theta = 292.62°$ and

$5-12i = = 13(\cos 292.62° + i\sin 292.62°)$

Step 2: Let $z^2 = 13(\cos 292.62° + i\sin 292.62°)$.

Then $z = \sqrt{13}\left(\cos\frac{292.62°+360°k}{2} + i\sin\frac{292.62°+360°k}{2}\right)$, $k = 0, 1$. (Two k-values for $n = 2$)

When $k = 0$, $z = \sqrt{13}\left(\cos\frac{292.62°}{2} + i\sin\frac{292.62°}{2}\right)$

$= \sqrt{13}(\cos 146.31° + i\sin 146.31°)$

$= \sqrt{13}(-.83 + .55i)$

$z = -2.99 + 1.98i$ (1)

When $k = 1$, $z = \sqrt{13}\left(\cos\frac{292.62°+360°(1)}{2} + i\sin\frac{292.62°+360°(1)}{2}\right)$

$z = \sqrt{13}\left(\cos\frac{292.62°+360°}{2} + i\sin\frac{292.62°+360°}{2}\right)$

$= \sqrt{13}(\cos 326.31° + i\sin 326.31°)$

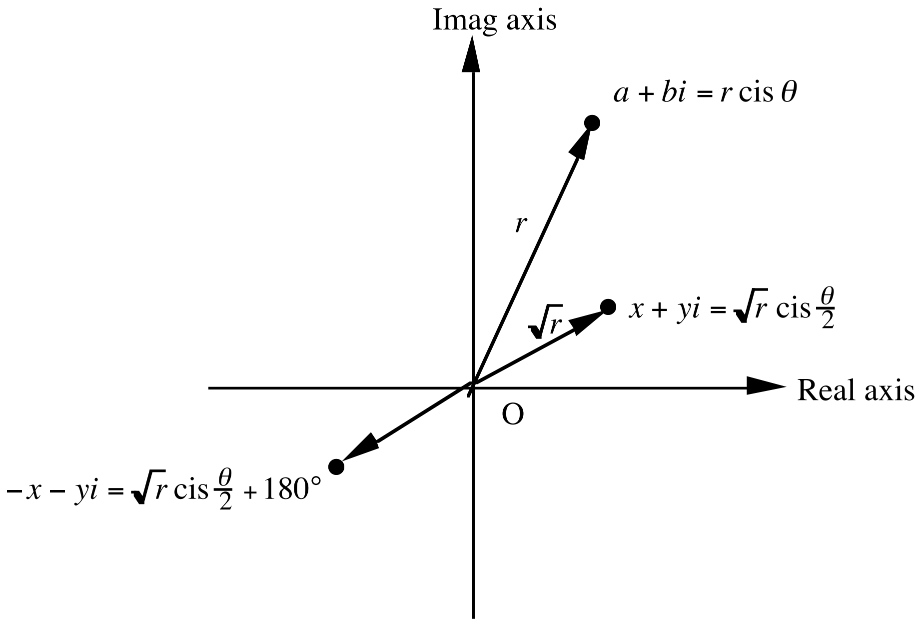
$= \sqrt{13}(.83 + i(-.55))$ ($\cos 326.31° = .83$; $\sin 326.31 = -.55$)

$= \sqrt{13}(.83 - .55i)$

$= 2.99 - 1.98i$ (2)

The square roots are $-2.99 + 1.98i$ and $2.99 - 1.98i$ (Combining the roots for $k = 0$ and $k = 1$)

Another Method of Finding Square Root (Short-Cut Method)



Let $x + yi$ be the square root of $a + bi$

Then $(x + yi)^2 = a + bi$ (1)

$x^2 - y^2 + 2xyi = a + bi$ (2)

Equating the real parts: $x^2 - y^2 = a$ (3)

Equating the imaginary parts: $2xy = b$ (4)

$\left(\sqrt{r}\right)^2 = x^2 + y^2$ (From figure)

$\therefore \quad r = x^2 + y^2$ (5)

Also, $r^2 = a^2 + b^2$ (From figure)

$r = \sqrt{a^2 + b^2}$ (6)

Equating (5) to (6),

$x^2 + y^2 = \sqrt{a^2 + b^2}$ (7)

We shall apply equations (3), (4), (6), and (7) to find the square root of a complex number, by redoing Example 3, above

Example 3 (Method 2) Find the square root of $5 - 12i$.

Step 1: Let $x + yi$ be the square root of $5 - 12i$.

From $r^2 = 5^2 + (-12)^2$,

$r = 13$ (1)

Applying $x^2 + y^2 = \sqrt{a^2 + b^2}$

$x^2 + y^2 = 13$ (2)

Step 2: $x^2 - y^2 + 2xy = 5 - 12i$

Equating the real parts: $x^2 - y^2 = 5$ (3)

Equating the imaginary parts: $2xy = -12$ (4)

We now solve for x and y using equations (2) , (3) and (4)

Step 3: By adding (2) and (3); we eliminate y^2.

$2x^2 = 18$

$x^2 = 9$ and $x = +3$ *or* -3

Step 4: We find y by substituting the x-values in equation (4)

When $x = 3$, $2(3)y = -12$ and from which $y = -2$.

When $x = -3$, $2(-3)y = -12$ and from which $y = 2$.

Thus, when $x = 3$, $y = -2$, and when $x = -3$, $y = 2$

Substituting these pairs (of values) in $x + yi$, the square roots are $3 - 2i$ and $3 + 2i$.

Plotting Roots in the Complex Plane

If we graph the roots, all the roots will be found to be equally spaced on a circle of radius, $\sqrt[n]{r}$ (or $r^{\frac{1}{n}}$), and the roots are $\frac{360}{n}$ degrees from one another. The first root is known as the principal nth root. The roots form the vertices of a regular n-sided polygon with its center at the origin.

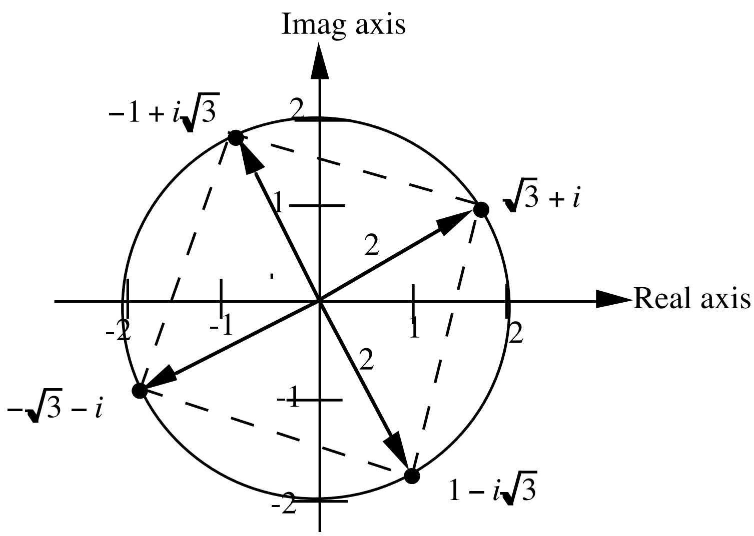


Figure: Graph of the roots for Example 1: $\sqrt{3} + i$; $-1 + i\sqrt{3}$; $-\sqrt{3} - i$; $1 - i\sqrt{3}$.

Lesson 45 Exercises

A Use De Moivre's theorem to evaluate the following:

1. $(2-3i)^4$; **2**. $(4+4i)^5$; **3**. $(-3+i)^6$; **4**. $(4\,cis\,30°)^5$; **5**. $(-i)^{11}$

Answers: **1**. $-119+120i$; **2**. $-4096-4096i$; **3**. $-352-934i$; **4**. $-886.81+512i$; **5**. i.

B Find the indicated roots.

1. The cube roots of 8; **2**. The fifth roots of 1; **3.** The fifth roots of 3;

4. The square roots of $2-3i$; **5.** The cube roots of $-1+i$

Find all the roots of the following: **6** $16x^4=-81$; **7**. $x^3+8=0$; **8.** $x^3+8i=0$

Answers:

1. $2,\ -1+i\sqrt{3},\ -1-i\sqrt{3}$; **2**. $1,\ 0.31+0.95i,\ -0.81+0.59i,\ -0.81-0.59i,\ .31-0.95i$;

3. $\sqrt[5]{3},\ \sqrt[5]{3}(0.31+0.95i),\ \sqrt[5]{3}(-0.81+0.59i),\ \sqrt[5]{3}(-0.81-59i),\sqrt[5]{3}(.31-0.95i)$;

4. $\sqrt[4]{13}(-0.88+0.47i),\sqrt[4]{13}(0.88-0.47i)$;

5. $\sqrt[6]{2}(0.71+0.71i),\ \sqrt[6]{2}(-0.96+0.26i),\ \sqrt[6]{2}(0.26-0.96i)$;

6. $1.061+\ 1.061i,\ -1.061+1.061i, -1.061-1.061i,\ 1.061-1.061i$;

7. $-2,\ 1-i\sqrt{3},\ 1+i\sqrt{3}$; **8.** $2i,\ -\sqrt{3}-i,\ \sqrt{3}-i$.

APPENDIX F

Lesson 46: Graphing Polar Coordinates
Lesson 47: Graphing Polar Equations

Introduction to Polar Coordinates

We use polar coordinates in the design of control systems in engineering. Some equations, say, in x and y which are fairly complex in the rectangular forms become relatively simple when changed to the equivalent polar forms. For example, let us consider the equation of the circle

given by $x^2 + y^2 = 9$ <------------rectangular form.

ln polar form, $r^2 = 9$ or simply $r = \pm 3$ <------------polar form (a circle of radius r)

Similarly, in polar form, the equation of the line $y = x$ is given simply by $\theta = 45°$ (for all r)

Lesson 46
Graphing Polar Coordinates

The Polar Coordinate System

As shown in Figures 1 and 2, the **rectangular coordinate system** has two axes, namely the x-axis and the y-axis. These two axes intersect at right angles at the origin which we label (0, 0).
A point P in the plane of this system has just one pair of coordinates (x, y) which is an ordered pair. Also an ordered pair of coordinates (x, y) represents only one point.

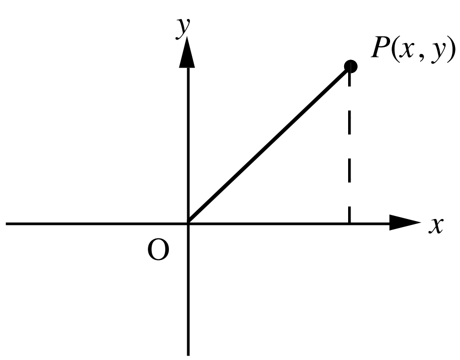


Figure 1: The rectangular coordinate system

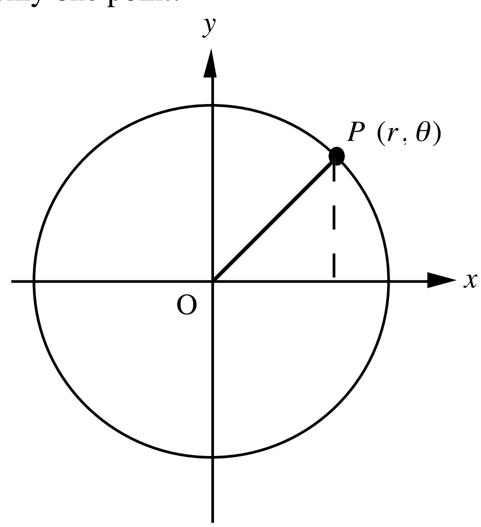


Figure 2: The polar coordinate System

Similarly, the **polar coordinate system** (Figure) has two axes, the x-axis (the **polar-axis** or 0°-axis) and the y-axis ($\frac{\pi}{2}$ or 90° axis) . These two axes intersect at right angles at the origin (or **the pole**) which we label $(0, \theta)$, where θ is any angle. A **g**iven **point** P in the polar coordinate system does **not** have just one pair of coordinates (in contrast to the coordinates in the rectangular coordinate system), but has an unlimited number of polar coordinate pairs. Each polar coordinate pair (r, θ) is ordered. However, we must note that a given **ordered pair** represents only one point.

Thus a given point has many names, but any of the possible names represents only one point, the given point.

A given point in the polar coordinate system has an unlimited number of coordinate pairs because different angular rotations to the same point are possible; and also, the radius can be generated in more than one assumed direction.

For the ordered pair (r, θ), r is the **directed distance** (may be positive or negative) from the origin (the pole) and θ, the **angle of rotation**, may be in degrees or in radians, and may be positive or negative. The angle is positive if measured counterclockwise but negative if measured clockwise. The angles are measured in the same way as done in trigonometry.

Although a point has an unlimited polar coordinates, we shall restrict θ so that $0 \le \theta \le 2\pi$. We then obtain what is called the **two primary representations** of the polar coordinates. The primary representations are $(r, \theta))$ and $(-r, \theta + \pi)$.

How to Plot Polar Coordinates

We shall now learn how to **locate a point** in the polar plane given the coordinates (r, θ) of the point. We shall plot on polar graph paper whenever it is available.

Step 1: Assuming we are **standing at the origin** (the pole) and facing in the positive *x*-direction, we **rotate** through the given angle θ counterclockwise if θ is positive; or clockwise if θ is negative. From this step, we go to Step 2 if *r* is positive but if *r* is negative, we go on to Step 3, below.

Step 2: Keeping the direction in which we are facing as a result of our rotation from Step 1, we **count** straight **forwards a distance** *r* units (*r* being positive) and then **stop**. We **label** this stopping point (r, θ) if θ is positive or $(r, -\theta)$ if θ is negative.
We then draw *r* with a **solid line**.

Step 3: If *r* is negative, then, keeping the direction in which we are facing as a result of rotation from Step 1, we **count** straight **backwards** a distance *r* units and then **stop**. We **label** this point $(-r, \theta)$ if θ is positive or $(-r, -\theta)$ if θ is negative. Finally, we draw this line segment *r* with a **broken** (or dotted) **line** to indicate that *r* is negative.

Example 1 Plot the point having the polar coordinates $(3, 60°)$

Solution

Step 1: Assuming you are at the origin and facing in the positive x-direction (Figure 1.), rotate through $60°$ counterclockwise (draw $60°$ counterclockwise since θ is positive).

Step 2: From the origin count 3 units forwards and stop. Place a dot here and label this stopping point $(3, 60°)$. If θ were in radians, we would label this point $(3, \frac{\pi}{3})$

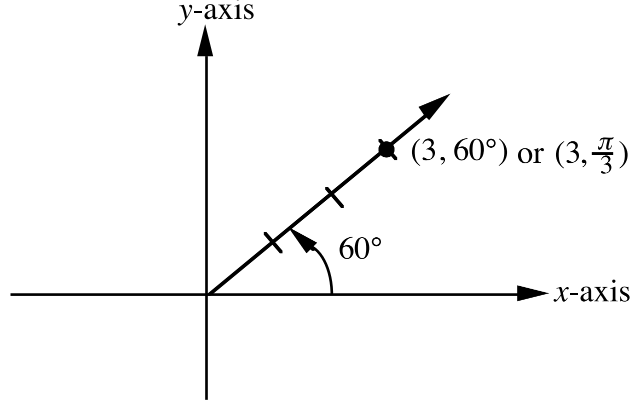


Figure 1 : The graph of the point $P(3, 60°)$

Example 2 Plot the point having the polar coordinates ($(3, -60°)$

Solution

Step 1: Since the angle is negative, we draw $60°$ clockwise (Figure 2)

Step 2: From the origin count 3 units forwards and stop. Place a dot here and label this stopping point $(3, -60°)$

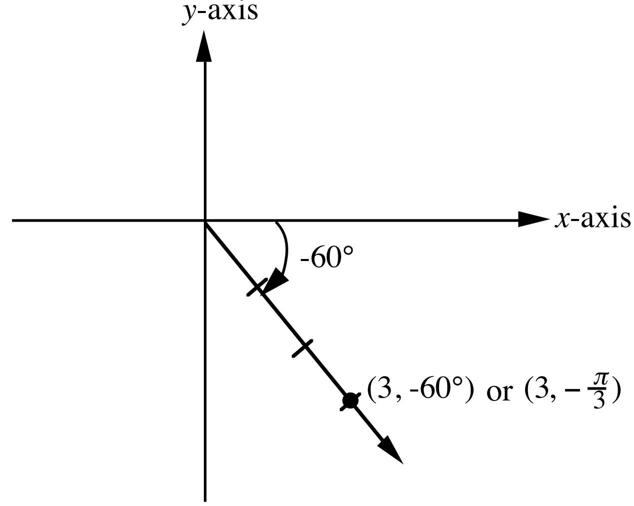


Figure 2 : The graph of the point $P(3, -60°)$

Example 3 Plot the point having the polar coordinates $(-3, 60°)$

Solution

Step 1: Assuming you are at the origin and facing in the positive *x*-direction (Figure.), rotate through $60°$ counterclockwise (draw $60°$ counterclockwise since θ is positive).

Step 2: Still at the origin and keeping the direction in which you are facing, count 3 units straight backwards along OQ and stop. Place a dot here, and label this point $(-3, 60°)$. Using a broken line draw the radius by connecting OQ.

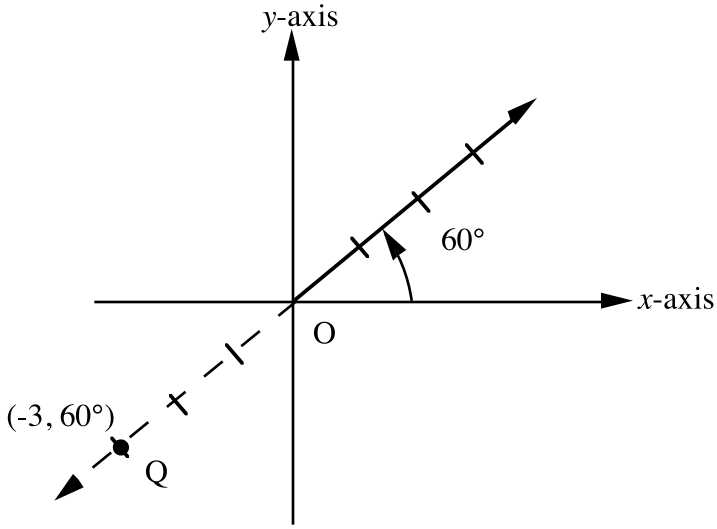


Figure :The graph of the point $Q(-3, 60°)$

Example 4 Plot the point having the polar coordinates $(-4, -60°)$

Step 1: Assuming you are at the origin and facing in the positive *x*-direction, rotate through $60°$ clockwise (since θ is negative). See Figure 3

Step 2: Still at the origin and keeping the direction in which you are facing as a result of rotation from Step 1, count 4 units straight backwards along OQ and stop. Place a dot here, and label this point $(-4, -60°)$.

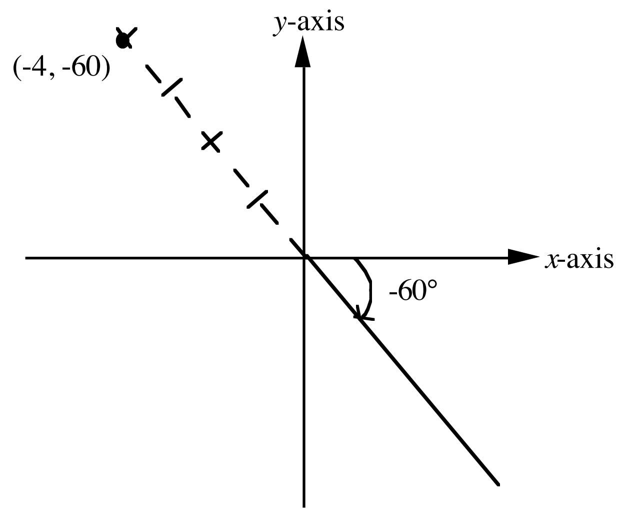


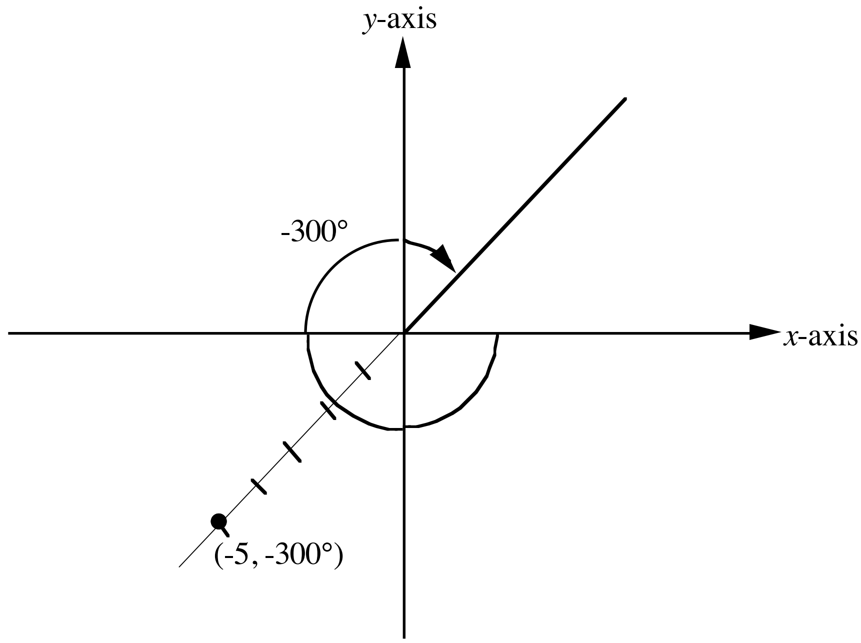
Figure 3 : The graph of the point $(-4, -60°)$

Example 5 Plot the point having the polar coordinates $(-5, -300°)$

Solution

Step 1: Rotate through $300°$ clockwise (since θ is negative).

Step 2: Count 5 units backwards and stop. Place a dot here and label this point $(-5, -300°)$.



Note: In plotting polar coordinates, we must note that the signs of the ordered pairs (r, θ) depend solely on **how the angles and distances are generated** and do **not** depend on the **quadrant** in which the point happens to be located, unlike the case of the rectangular coordinate system, in which the quadrant determines the signs of the coordinates.

The coordinates (r, θ) and, $(-r, \theta + \pi)$ represent the same point. These two coordinate pairs are the two primary representations of the same given point. There are infinitely many other representations each of which is obtained by adding 2π or (or -2π) successively to the primary representation.

Other representations of the same point: are

$(r, \theta + 2\pi)$ and $(-r, \theta + 3\pi)$.

$(r, \theta - 2\pi)$ and $(-r, \theta - \pi)$.

Determining if a given point is on a curve, given the polar equation

Definition: A given point (except the pole) is on a polar form curve if and only if at least one of the primary representations of the point satisfies the given equation.

Thus if we check for one primary representation and it does not satisfy the given equation, we must also check for the other primary representation.

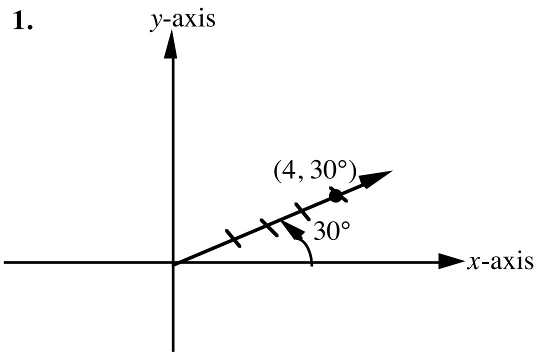
To check for a point, we substitute the given coordinates in the equation to determine if the left-hand side equals the right-hand side of the equation. If the LHS equals the RHS the point is on the curve, otherwise, it is not on the curve

Lesson 46 Exercises

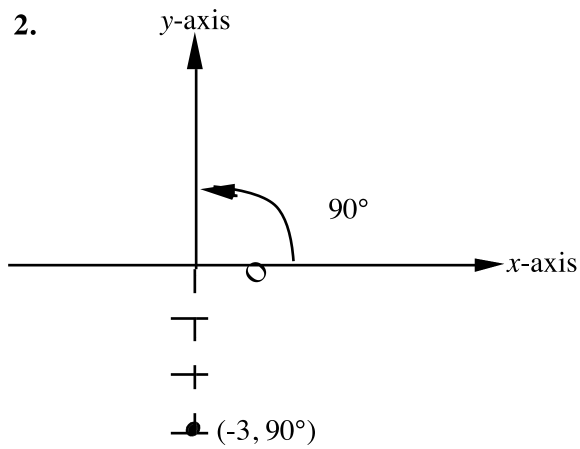
Plot the following polar coordinates:

1. $(4, 30°)$; **2**. $(-3, 90°)$; **3**. $(-3, \frac{\pi}{3})$; **4**. $(-6, -30°)$; **5**. $(-4, -400°)$

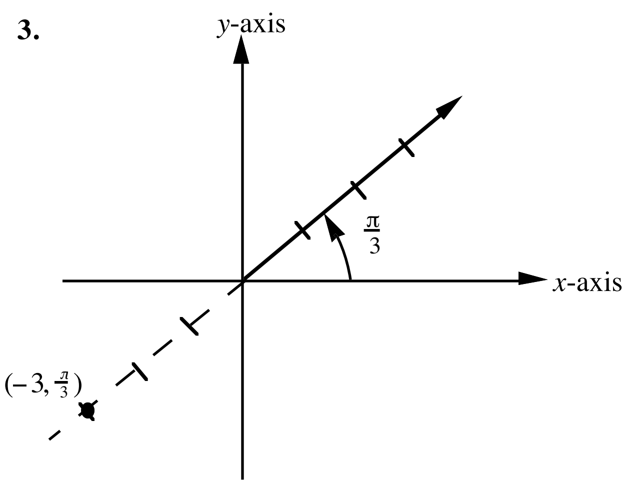
1.



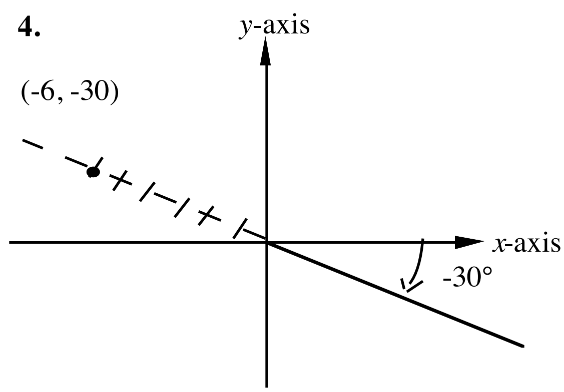
2.



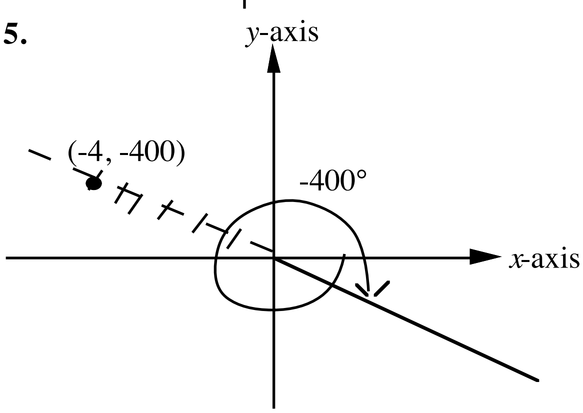
3.



4.



5.

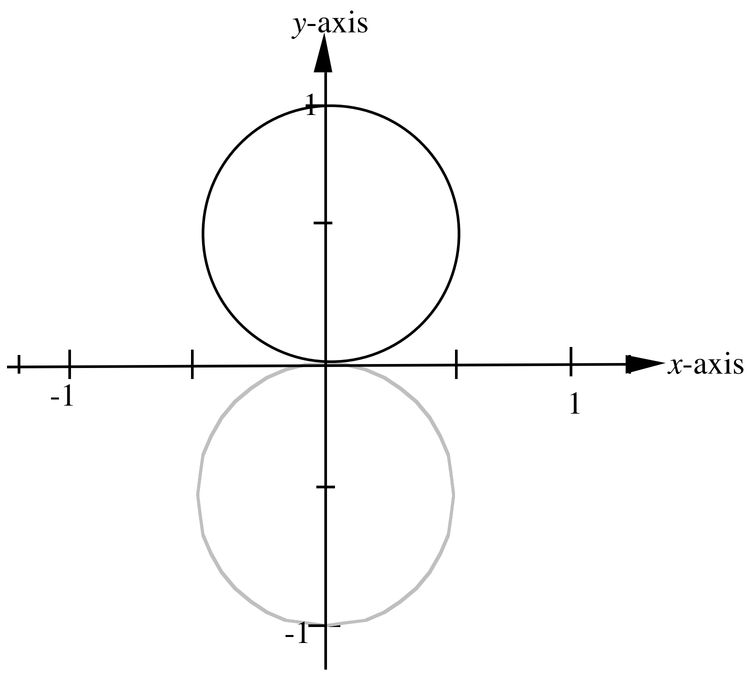


Lesson 47
Graphing Polar Equations

In general, we can sketch the graphs of polar equations by constructing tables for r and θ and plotting points. Consider a table for the polar equation $r = \sin\theta$ (See Table 1). It can be observed from the table that after π radians or 180° the values of r begin to duplicate. We can therefore take advantage of such duplication when sketching polar graphs. We can do this by testing to determine if the graph is symmetric about (a) the x-axis, (b) the y-axis, and (c) the origin.

Table 1

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	240°	225°	270°	300°	315°	330°	360°
$r = \sin\theta$	0	.5	.71	.87	-1	.87	.71	.5	0	-.5	-.87	-.71	-1	-.87	-.71	-.5	0



Rules for testing for symmetry about the *x*-axis

Step 1: Replace (r, θ) by either $(r, -\theta)$, or $(-r, -\theta + \pi)$ and simplify If the equation remains unchanged on simplifying then the curve is symmetric about the x-axis.

Step 2: if $(r, -\theta)$ fails the test, we must check for $(-r, -\theta + \pi)$. If both $(r, -\theta)$ and $(-r, -\theta + \pi)$ fail the test, then the curve is **not** symmetric about the x-axis However if either $(r, -\theta)$ or $(-r, -\theta + \pi)$ passes the test first, then we do not have to test for the other.

Although we can test for symmetry before sketching polar curves , for common practical purposes and to save time, we may not test for symmetry for the common polar graphs. We shall familiarize ourselves with or memorize the properties of the common or simple polar curves (the same as we familiarized ourselves with the simple trigonometric functions such as $y = \sin x$.). When we meet an unfamiliar or a more complicated polar equation, then we shall test for symmetry We shall now discuss and sketch some common polar graphs. In the future, we should be able to identify these graphs with the corresponding equations and vice versa.

The following trigonometric identities will be useful when testing for symmetry and graphing polar equations:

1. $\cos(-\theta) = \cos\theta$

2. $\sin(-\theta) = -\sin\theta$

3. $\tan(-\theta) = -\tan\theta$

4. $\cos(\pi - \theta) = -\cos\theta$

5. $\sin(\pi - \theta) = \sin\theta$

6. $\cos(\pi + \theta) = -\cos\theta$

7. $\sin(\pi + \theta) = -\sin\theta$

Also, $\cos(\theta \pm 2\pi) = \cos\theta$; $\sin(\theta \pm 2\pi) = \sin\theta$

The Rose Curves: The equations of the rose curves are

$r = a\sin n\theta$ (where n ia a positive integer)

$r = a\cos n\theta$

If n is odd each curve has n leaves but has $2n$ leaves if n is even.

Example 1 If $n = 1$ (odd number), the sine equation, $r = a\sin n\theta$ becomes $r = a\sin\theta$ and the curve has one leaf (petal). The leaf is circular and has the y-axis as the axis of symmetry (Figure 1).

Example: $r = 2\sin\theta$ (where $a = 2$,)

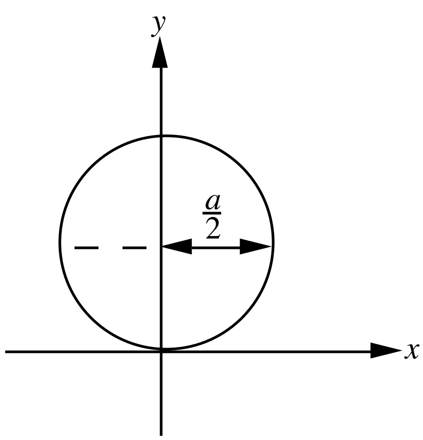


Figure 1 : $r = a\sin\theta$

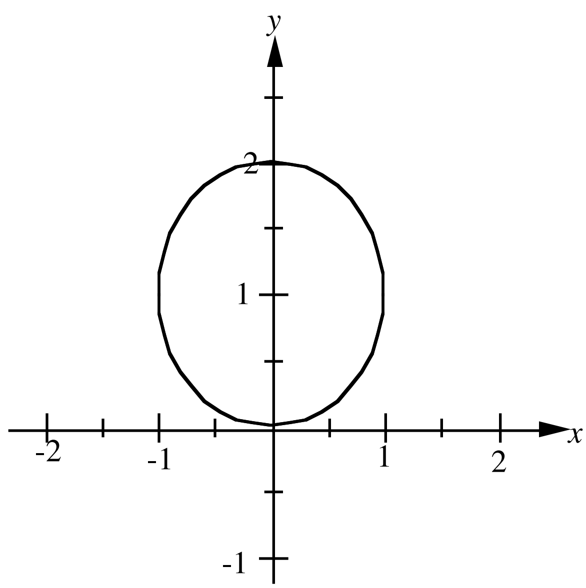


Figure 2 : Graph of $r = 2\sin\theta$

Example 2 If $n = 1$, the cosine equation becomes $r = a\cos\theta$. This curve also like that of the sine has one circular leaf, but the axis of symmetry is the x-axis .
Example: $r = 2\cos\theta$ (where $a = 2$,)

-

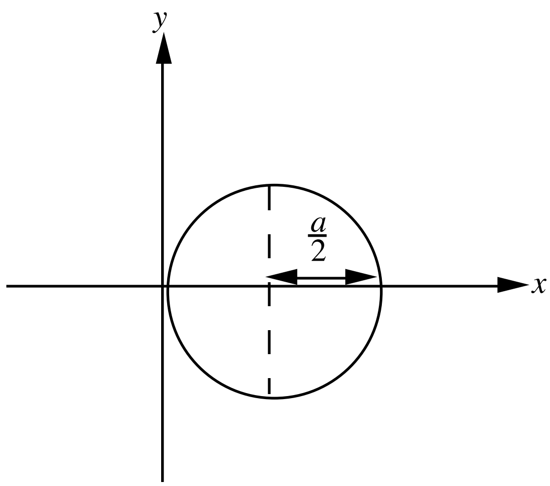


Figure Graph of $r = a\cos\theta$

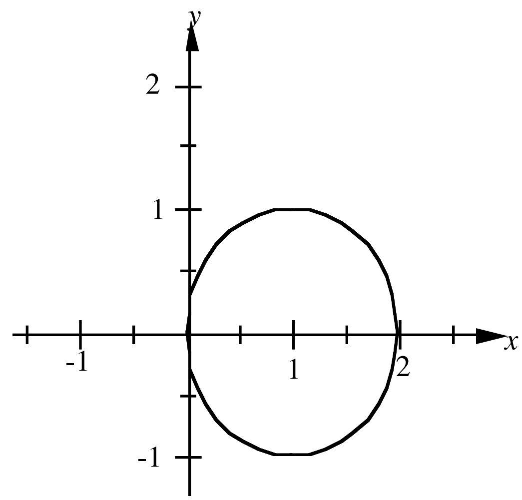


Figure: Graph of $r = 2\cos\theta$

Example 3 Sketch the polar graph for $r = -a\sin\theta$

Solution The graph is the graph of $r = -a\sin\theta$ reflected about the y-axis.
Compare Figure and

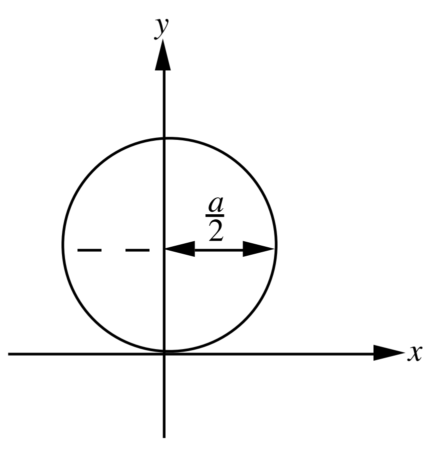


Figure : Graph of $r = a\sin\theta$

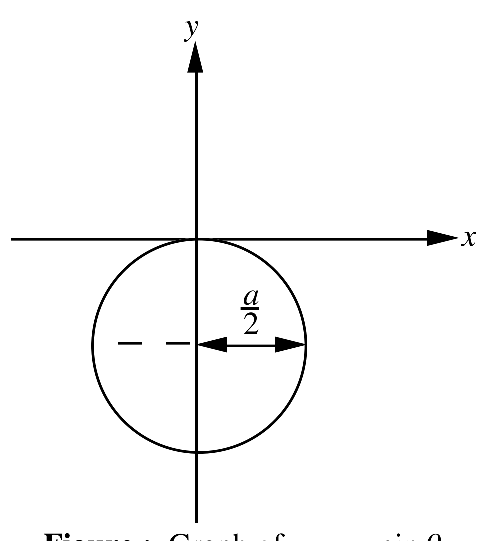


Figure : Graph of $r = -a\sin\theta$

Example 4 Sketch the polar graph for $r = -a\cos\theta$.

Solution The graph of $r = -a\cos\theta$ is the graph of $r = a\cos\theta$ reflected reflected about the y-axis. Compare Figure and

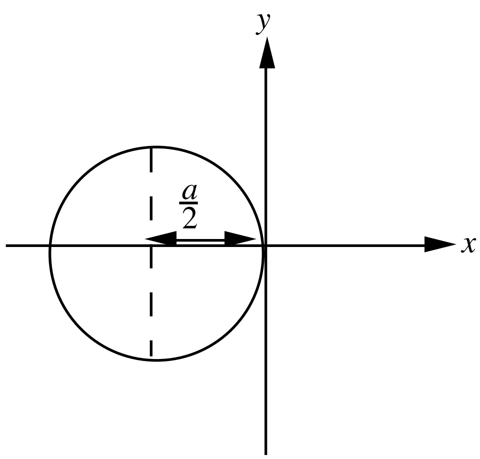


Figure: $r = -a\cos\theta$

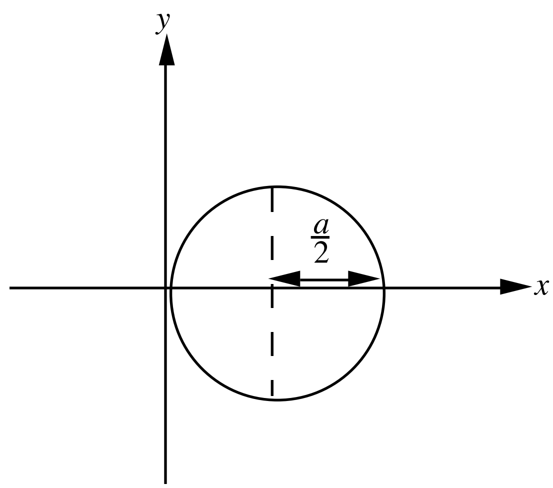
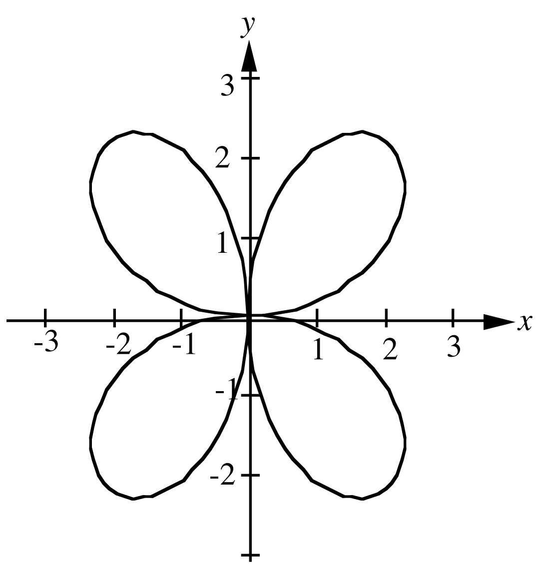


Figure: $r = a\cos\theta$

Example 5 Sketch the graph of $r = 3\sin 2\theta$
Solution

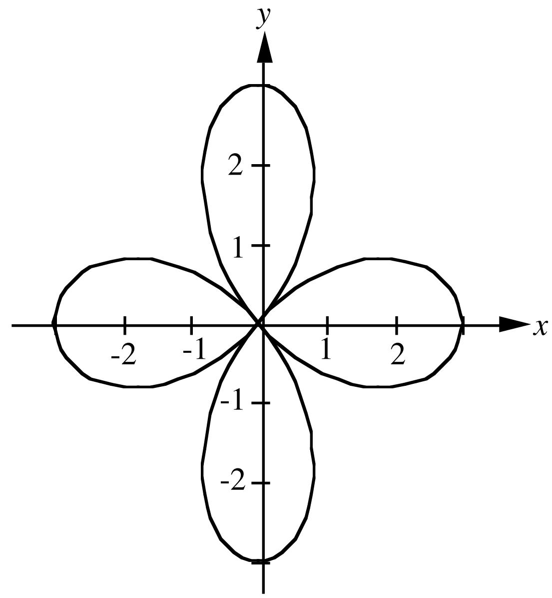
Since n is even ($n = 2$), there are 2(2) or 4 leaves or petals. There are $\frac{360°}{4} = 90°$ between the axes of any two leaves. The first leave is in the first quadrant. The leaves are symmetric about the lines $y = x$, and $y = -x$ (Figure 203)



Example 6 Sketch the graph of $r = 3\cos 2\theta$

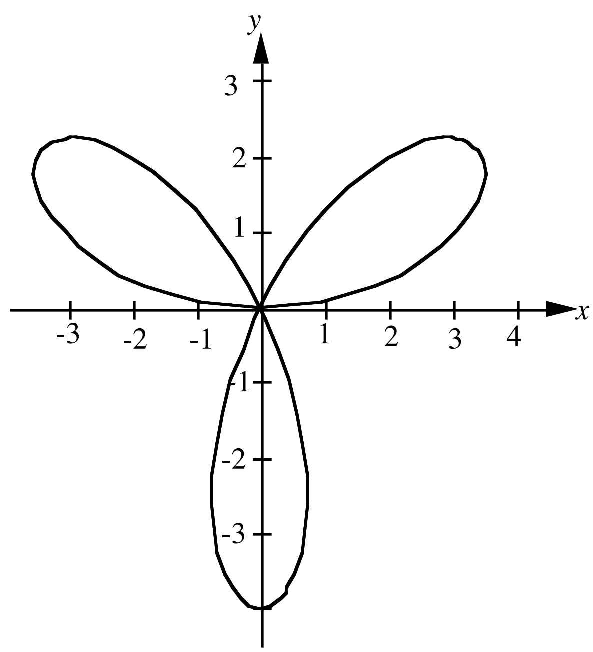
Solution

Since $n = 2$, there are 2(2) or 4 leaves. The are $\frac{360°}{4} = 90°$ between the axes of any two leaves (as in Example 5 above). However, the axes of symmetry are the *x*- and *y*-axes (Figure).



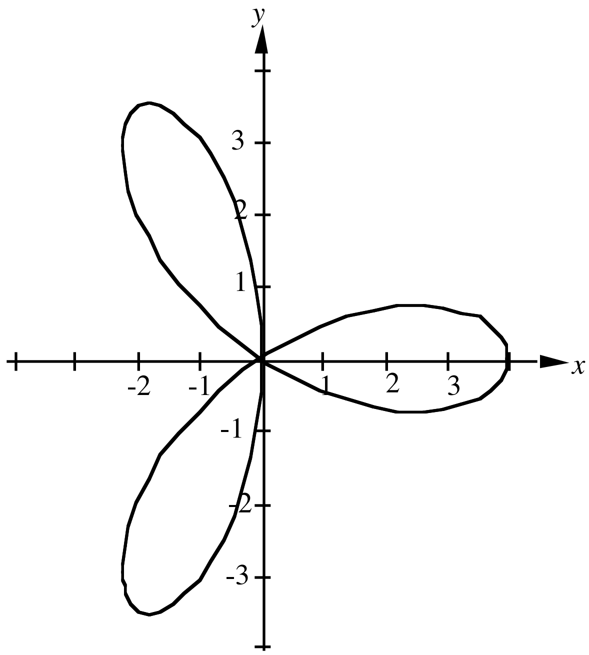
Example 7 Sketch the graph of $r = 4\sin 3\theta$

Solution Since n is odd and $n = 3$, there are 3 leaves. There are also $\frac{360°}{3} = 120°$ between the axes of any two leaves. The first leave is in the first quadrant and it is symmetric about the line $\frac{\pi}{6}$ (Figure).



Example 8 Sketch the graph of $r = 4\cos 3\theta$

Solution The graph is similar to that in Example 7, except that there is a difference in the location of the petal (leaf) axes (Figure)



Other polar Graphs

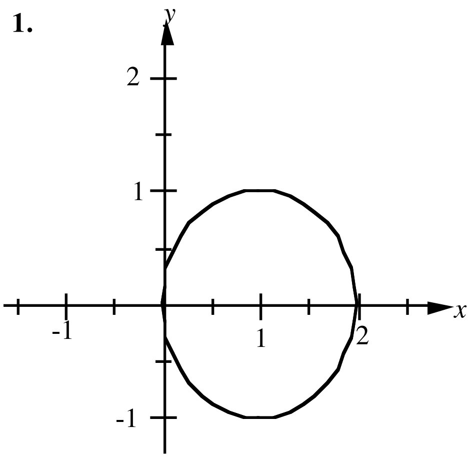
The graphs of the Limacon $r = a \pm b\sin\theta$; $r = a \pm b\cos\theta$; $r^2 = a^2\sin 2\theta$; $r^2 = a^2\cos 2\theta$

Lesson 47 Exercises

Identify (e.g., rose curve) and sketch the graph of each equation.

1. $r = 2\cos\theta$; **2**. $r = 5\sin\theta$; **3.** $r = -4\sin\theta$; **4**. $r = 4\sin 3\theta$; **5**. $r = 5\cos 3\theta$;
6. $r = 4$; **7**. $r = 2\theta$; **8**. $r = 4 - 3\sin\theta$; **9**. $r^2 = 5\cos 2\theta$; **10**. $r = 3 - 3\cos 3\theta$;
11. $r = e^{2\theta}$; **12**. $\theta = \frac{\pi}{6}$.

1.



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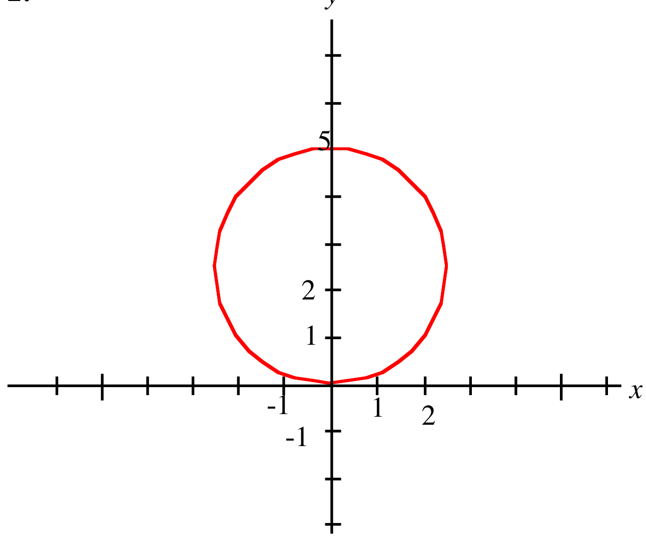

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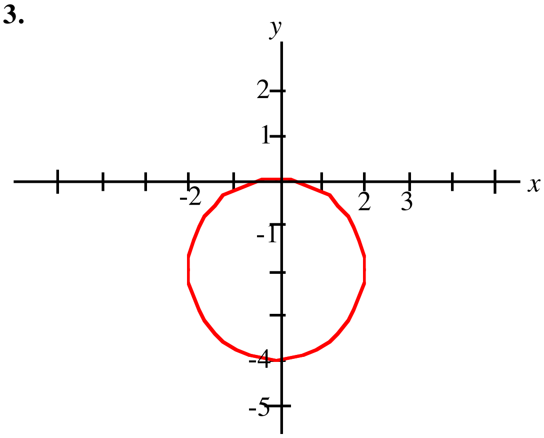

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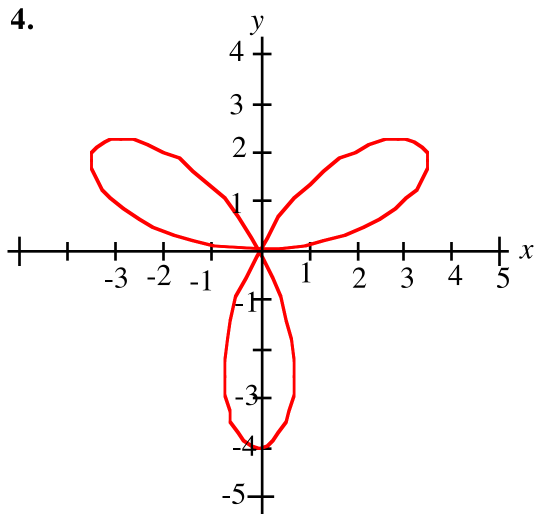

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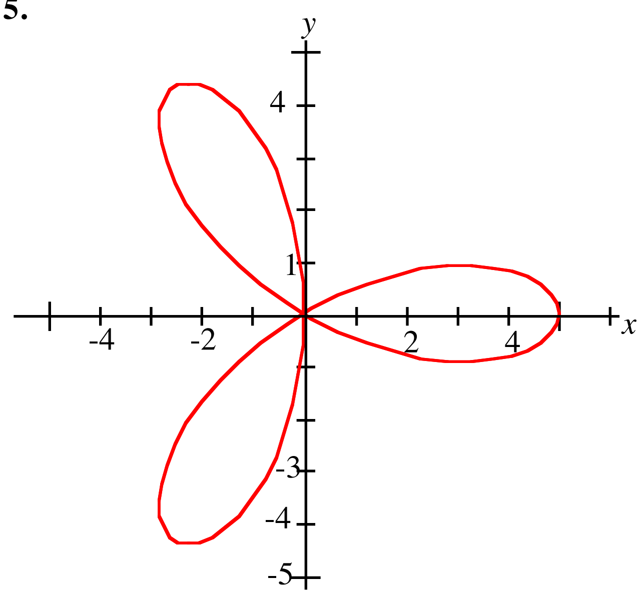

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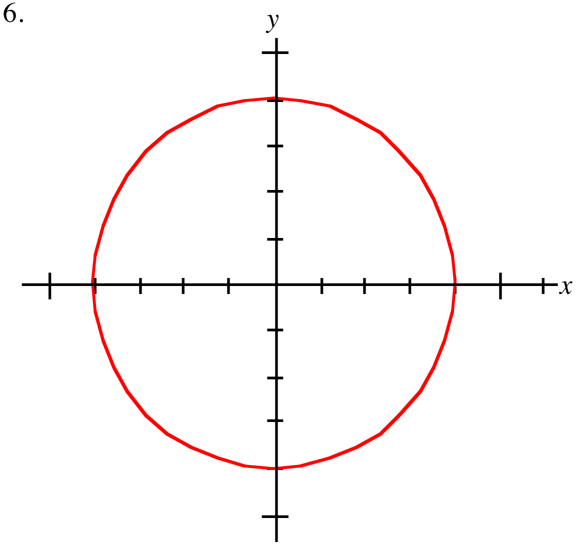

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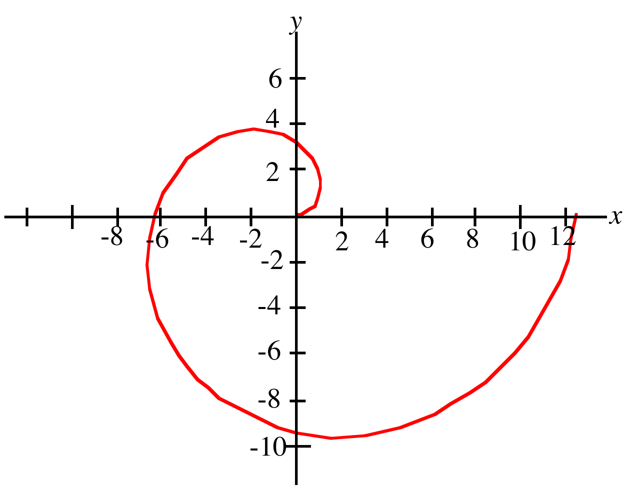

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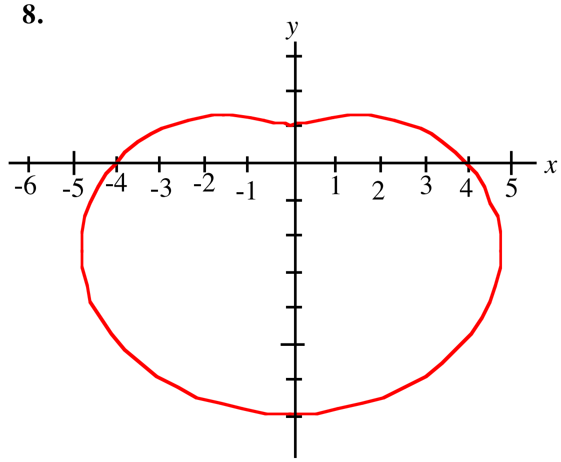

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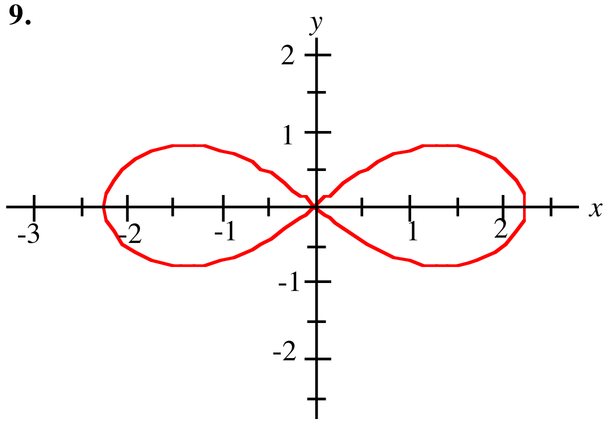
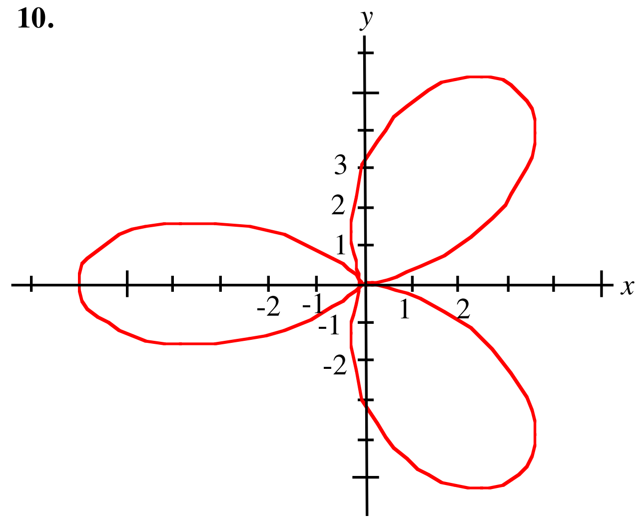
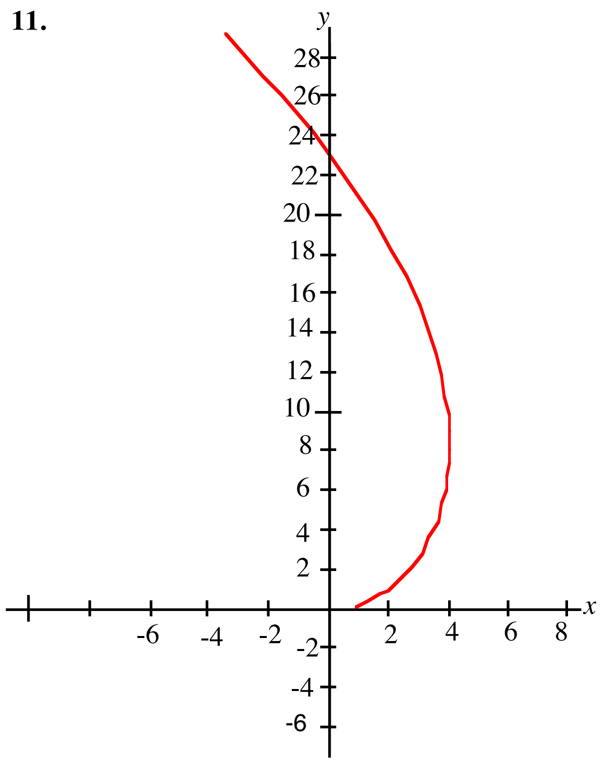

y
2
1
-1
-2
x
-3
-2
-1
1
2

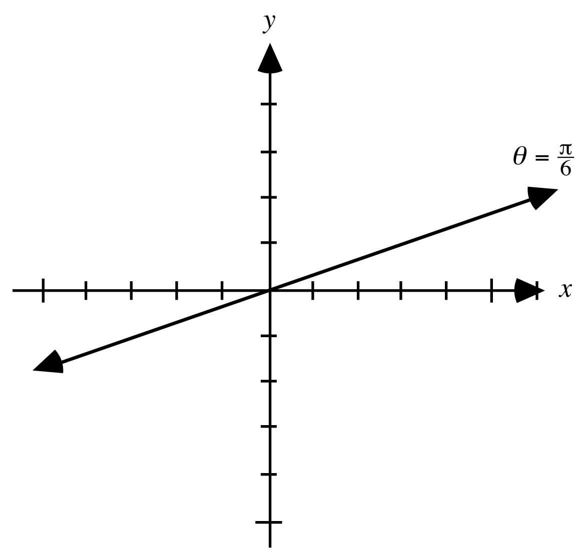
10.


y
3
2
1
-1
-2
x
-2
-1
1
2

11.


y
28
26
24
22
20
18
16
14
12
10
8
6
4
2
-2
-4
-6
x
-6
-4
-2
2
4
6
8

12


y
x
$\theta = \frac{\pi}{6}$